ON A FAMILY OF SEMIGROUP CONGRUENCES

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OF
SEMIGROUP CONGRUENCES

by

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ABSTRACT

We introduce in this thesis a new family of semigroup congruences, and we set out to prove that it is worth studying them for the following very important reasons:

(a) that it provides an alternative way of studying algebraic structures of semigroups, thus shedding new light over semigroup structures already known, and it also provides new information about other structures not formerly understood;

(b) that it is useful for constructing new semigroups, hence producing new and interesting classes of semigroups from known classes; and

(c) that it is useful for classifying semigroups, particularly in describing lattices formed by semigroup species such as varieties, pseudovarieties, existence varieties etc.

This interesting family of congruences is described as follows: for any semigroup S, and any ordered pair (n,m) of non-negative integers, define

$$\theta(n,m) = \{(a,b): uav = ubv, \text{ for all } u \in S^n \text{ and } v \in S^m\},$$

and we make the convention that $S^1 = S$ and that S^0 denotes the set containing only the empty word. The particular cases $\theta(0,1)$, $\theta(1,0)$ and $\theta(0,0)$ were considered by the author in his M.Sc. thesis (1991). In fact, one can recognise $\theta(1,0)$ to be the well known kernel of the *right regular representation* of S. It turns out that if S is *reductive* (for example, if S is a monoid), then $\theta(i,j)$ is equal to $\theta(0,0)$ — the identity relation on S, for every (i,j).

After developing the tools required for the latter part of the thesis in Chapters 0-2, in Chapter 3 we introduce a new class of semigroups — the class of all *structurally regular semigroups*. Making use of a new Mal'tsev-type product, in Chapters 4,5,6 and 7, we describe the lattices formed by certain varieties of structurally regular semigroups.

Many interesting open problems are posed throughout the thesis, and brief literature reviews are inserted in the text where appropriate.

DECLARATION

I, **Samuel Joseph Lyambian Kopamu**, hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for a higher degree.

Signed . . . . Date . . 26/April/1996

I was admitted to the faculty of Science in the University of St. Andrews under the ordinance General No. 12 in November 1993, and as a candidate for PhD in September 1994.

Signed . Date . . 26/April/1996

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CERTIFICATION

I certify that the candidate has fulfilled the conditions of Resolutions and Regulations appropriate to the degree of PhD.

Signature of Supervisor .... Date ....

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Dedication

This thesis is dedicated firstly to the memory of my loving mother

Esther Tombolyon Nangule,

who passed away on the 27th of December 1990;

and secondly,

to the

Kokopes and *Porealins* of Mukuris and Mulisoas Villages

who will never know anything about the fascinating algebraic theory of Semigroups — the very thing that has captivated my curiosity all these years, causing me to leave home, and wander across the ocean to the other side of the world;

and thirdly,

I toast a glass of water to the prospect of a brighter future and prosperity for my motherland — Papua New Guinea.

CONTENTS

CHAPTER 0

PRELIMINARIES

For an introduction to the algebraic theory of semigroups, we refer the reader to the following classical texts: Clifford and Preston (1961) and (1967), Howie (1976) and (1995), Petrich (1984) and Higgins (1992). In this preliminary chapter we state only the fundamental concepts and results that will be of use throughout the thesis.

0:1 FUNDAMENTAL CONCEPTS

A semigroup $(S,\circ)$ is a set S with an associative binary operation $\circ$, which we usually denote by juxtaposition; that is, whenever the operation $\circ$ is understood, we write $a\circ b$ simply as ab for any a,b in S. The associativity of this binary operation allows us to write the products of any length without the need of brackets. A semigroup with an identity element is called a *monoid*. By *adjoining an identity element*, 1, to a semigroup S (whether or not it already has an identity) we mean the following: for all a in $S \cup \{1\}$, define $1a = a1 = a$, and the semigroup so produced is denoted by $S^{(1)}$. Similarly, we adjoin a zero element, 0, to S by defining $a0 = 0 = 0a$ for all a in $S \cup \{0\}$; and the semigroup so constructed is denoted by $S^{(0)}$.

An element of a semigroup S satisfying the condition $e^2 = e$ is called an *idempotent*; and we denote by E(S) the set of all such elements in S. Semigroups consisting entirely of idempotents are called *bands*.

A *homomorphism* is a mapping $\phi: S \rightarrow T$ from a semigroup S into T such that $(ab)\phi = (a\phi)(b\phi)$ for all a,b in S. An *isomorphism* is a homomorphism that is both one to one and onto. An *endomorphism* [*automorphism*] is a homomorphism [isomorphism] from a semigroup into [onto] itself. A homomorphism ϕ from a semigroup S into itself is called a *retractive endomorphism* if $(x\phi)\phi = x\phi$ for all x in S. A subset T of a semigroup S is called a *subsemigroup* if T is closed under the binary operation on S. The *direct product* of semigroups A and B consists of the cartesian product $A \times B$ with the binary operation defined pointwise.

An important example of a semigroup is the *full transformation semigroup* $\mathcal{T}(X)$ on a non empty set X, which consists of all maps from X into itself under composition of functions. Any homomorphism from a semigroup into some $\mathcal{T}(X)$ is often called a *representation*. The following is called the *right regular representation* of S: define $a \mapsto \rho_a$, where the map $\rho_a : S \to S$, is defined by $x \mapsto xa$. The map is not one to one if S is not right reductive (see below), but we may replace S by S^1 and the resulting representation is then one to one, and is called the *extended right regular representation* of S (see Clifford and Preston (1961)).

A semigroup S is said to be *left reductive* if the following relation reduces to the identity relation:

$$\theta(1,0) = \{(x,y): \ sx = sy \text{ for } s \text{ in } S\}.$$

Monoids and groups are examples of such semigroups. The concept of *right reductivity* is defined by duality. We simply say a semigroup is *reductive* if it is both left and right reductive. Every [left,right] reductive semigroup S can be embedded into the semigroup $\mathcal{T}(S)$ under the [right,left] regular representation, as pointed out above.

Let A be a subset of a semigroup S. By the *subsemigroup generated* by A, and denoted by $\langle A \rangle$, we mean the smallest subsemigroup of S containing A. If $A = \{a\}$, a singleton set, then we speak of the *monogenic* subsemigroup generated by a. Then either $\langle a \rangle$ is isomorphic to the set of all natural numbers under addition, or there exist positive integers r,m such that

$$\langle a \rangle = \{a, a^2, a^3, \ldots, a^r, \ldots, a^{r+m-1}\} \text{ with } K_a = \{a^r, a^{r+1}, \ldots, a^{r+m-1}\},$$

a cyclic group of order m. The integers r and m will be called the *index* and the *period* of $\langle a \rangle$.

A *relation* ρ on a semigroup S is a subset of the Cartesian product $S \times S = \{(a,b) : \ a,b \in S\}$; and an *equivalence relation* is a relation that has the following additional properties:

(i) For all $x \in S$, $(x,x) \in \rho$ (reflexive)

(ii) If $(x,y) \in \rho$ then $(y,x) \in \rho$ (symmetric)

(iii) If $(x,y), (y,z) \in \rho$ then $(x,z) \in \rho$ (transitive).

An equivalence relation that is compatible with the multiplication on S in the following sense is called a *congruence*: for any $(x,y) \in \rho$ and any $a \in S$, we have $(ax,ay), (xa,ya) \in \rho$, in which case S/ρ is a semigroup with multiplication $(a\rho)(b\rho) = (ab)\rho$.

The five *Green's relations* defined below provide important tools in the algebraic structure theory of semigroups. They play no role in group theory since they all coincide with the universal equivalence relation. Let S be a semigroup and define:

(0:1.1) $a\mathcal{R}b$ if and only if $aS^1 = bS^1$. Equivalently, $a\mathcal{R}b$ if and only if there exists x,y in S^1 such that $ax = b$ and $by = a$.

(0:1.2) $a\mathcal{L}b$ if and only if $S^1a = S^1b$. Equivalently, $a\mathcal{L}b$ if and only if there exists x,y in S^1 such that $xa = b$ and $yb = a$.

(0:1.3) $\mathcal{D}$ the least equivalence relation containing $\mathcal{R}$ and $\mathcal{L}$. As shown by Green (1951), $a\mathcal{D}b$ if and only if there exists c in S such that $a\mathcal{L}c$ and $c\mathcal{R}b$; or equivalently, there exists d if S such that $a\mathcal{R}d$ and $d\mathcal{L}b$.

(0:1.4) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$

(0:1.5) $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$

For any a in S we shall denote the $\mathcal{L}$-class containing a in S by L_a, and in the same way define R_a, H_a, D_a, and J_a.

A *partial order* $\leq$ on a semigroup S is a relation which satisfies the reflexivity and transitivity conditions together with the additional property of *anti-symmetry*. A relation $\leq$ is said to be anti-symmetric if for any x and y, $x \leq y$ and $y \leq x$ implies $x = y$. A *natural partial ordering* is known to exist on the set of all idempotent elements of a semigroup as follows: $e \leq f$ if and only if $ef = fe = e$.

A subset I of a semigroup S is said to form an *ideal* if both $SI = \{si: s\in S, i \in I\}$ and $IS = \{is: i\in I, s\in S\}$ are contained in I, in which case I forms a subsemigroup and we say that S is an *ideal extension* of I by the quotient S/I^*, under the *Rees congruence* $I^* = \{(a,b): a,b\in I\} \cup \{(a,a): a\in S\backslash I\}$. In particular, if there exists a retractive endomorphism from S onto the ideal I, then S is called a *retract extension* of I by S/I^* with respect to ϕ.

0.2 REGULAR SEMIGROUPS

A semigroup is said to be regular if for each element a there exists an element b such that $aba = a$. Then the element $a' = bab$ is called an *inverse* of a and has the property that $a'aa' = a'$ and $aa'a = a$. The set of all inverses of a is denoted by $V(a) = \{b: bab = b, aba = a\}$.

Theorem 0:2.1 (Lemma II:4.7 of Howie (1976)) *Let a,b be regular elements of a semigroup S. Then*

(i) $a\mathcal{L}b$ if and only if for some [for all] $a' \in V(a)$ there exists $b' \in V(b)$ such that $a'a = b'b$.

(ii) $a\mathcal{R}b$ if and only if for some [for all] $a' \in V(a)$ there exists $b' \in V(b)$ such that $aa' = bb'$;

(iii) $a\mathcal{H}b$ if and only if for some [for all] $a' \in V(a)$ there exists $b' \in V(b)$ such that $a'a = b'b$ and $aa' = bb'$. ❑

Theorem 0:2.2 (Lallement's Lemma) *Let S be a regular semigroup and ϕ be a homomorphism from S onto T. Then for each idempotent element f of T there exists an idempotent e in S such that $e\phi = f$.* ❑

Of course, any homomorphic image of a regular semigroup is regular. The next result is a generalisation by T.E. Hall of the one above.

Theorem 0:2.3 *Let S be a regular semigroup and ϕ be a homomorphism from S onto T. Then for each element a of T, and any inverse a' of a, there exists an element b and an inverse b' of b in S such $b\phi = a$ and $b'\phi = a'$.* ❑

A regular semigroup is called *orthodox* if the set E(S) of all its idempotents form a subsemigroup. The following characterisation of orthodox semigroup is due to Reilly and Scheiblich (1967).

Theorem 0:2.4 *If S is a regular semigroup, then the following statements are equivalent:*

(A) S is orthodox

(B) for any a,b in S, if a' is an inverse of a and b' is an inverse of b, Then $b'a'$ is an inverse of ab;

(C) if e is an idempotent then every inverse of e is idempotent. ❑

Inverse semigroups form an important class of orthodox semigroups. These are just those regular semigroup with unique inverses. We shall denote by a^{-1} the unique inverse of a. The following result is a characterisation of inverse semigroups.

Theorem 0:2.5 *The following statements about a semigroup S are equivalent.*

(a) S is an inverse semigroup

(b) S is regular and its idempotent elements commute

(c) each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains a unique idempotent;

(e) each principal left ideal and each principal right ideal of S contains a unique idempotent generator. ❑

0:3 SYSTEMS OF CLASSIFICATION FOR SEMIGROUPS

Let X be a set, and let X^+ be the free semigroup on X, the semigroup of all non-empty words constructed from letters in X. An *identity*

$$P(x_1,x_2, \ldots ,x_k) = Q(x_1,x_2, \ldots ,x_t)$$

is the formal equality of the two words $P(x_1,x_2, \ldots ,x_k)$ and $Q(x_1,x_2, \ldots ,x_t)$. A semigroup S is said to *satisfy* an identity if for any substitution by elements of S for the letters in the words forming the identity, the two resulting elements are equal. Given a family $\mathcal{F}$ of identities, the *equational class* determined by $\mathcal{F}$ is the class of semigroups given by

$$\{S: S \text{ satisfies every identity in } F\}.$$

A semigroup *variety* is a class of semigroups closed under homomorphic images, arbitrary direct products and subsemigroups. From the result of Birkhoff (see Grätzer (1968)) we have that each semigroup variety is an equational class determined by some family of identities. More recently, certain related ideas have been introduced, and Birkhoff-type theorems have been proved in each of these cases. Some examples are the following cases: inverse semigroups and completely regular semigroups considered as unary semigroups (see Petrich (1984)); existence varieties of regular semigroups (see Hall (1989) and (1991)); pseudovarieties of finite (monoids) semigroups (see Eilenberg and Schutzenberger (1976)); and generalised varieties (Ash (1985)). What all these different notions have in common is the closure property under homomorphic images. In this thesis, semigroup classes that possess this property are referred to as *species* of semigroups. In the language of Cohn (1965), a species is a *co-hereditary* class.

In the remainder of this section, we will summarise briefly the different types of variety, showing how they relate to each other.

We begin with the concepts of *pseudovarieties* and *generalised varieties*. The concept of pseudovariety was first introduced by Eilenberg and Schutzenberger in 1976 within the context of finite semigroups. Later, in 1985, C.J. Ash introduced the notion of generalised varieties within the context of universal algebra, and extended the concept of pseudovarieties to cover finite algebras as well.

A family $\{H_i\}$ of sets is said to be *directed* if for all $A,B \in \{H_i\}$ there exists $C \in \{H_i\}$ such that $A \subseteq C$ and $B \subseteq C$. For any class $\mathcal{K}$ of semigroups, $H(\mathcal{K})$, $S(\mathcal{K})$, $P_f(\mathcal{K})$, $P(\mathcal{K})$ and $Pow(\mathcal{K})$ will denote, respectively, the classes of homomorphic images, subsemigroups, finite direct products, arbitrary direct products, and arbitrary powers of members of $\mathcal{K}$. Denote by E the set of all semigroup identities; and define a *filter* F over E to be a family of subsets of E with the property that F is closed under supersets and finite intersections. For each class A of semigroups, id(A) denotes the set of all semigroup identities satisfied by members of A.

The following result is a paraphrased version of a universal algebraic result.

Theorem 0:3.1 (Ash (1985)) *The following conditions on a class K are equivalent.*

(i) K is closed under $H(K)$, $S(K)$, $Pf(K)$ and $Pow(K)$

(ii) $K = HSPfPow(K)$

(iii) K is a union of a directed family of varieties

(iv) there exists a filter F over E such that, for any semigroup A,

$$A \in K \Leftrightarrow Id(A) \in F. \qquad \square$$

A class $\mathcal{K}$ of semigroups is said to be a *generalised variety* (Ash (1985)) if it satisfies the equivalent conditions of Theorem 0:3.1. A class $\mathcal{K}$ is said to be a *pseudovariety* if and only if it is closed under $H(\mathcal{K})$, $S(\mathcal{K})$, and $P_f(\mathcal{K})$; and we denote by $\mathcal{Fin}$ the pseudovariety of all finite semigroups.

Theorem 0:3.2 (Ash (1985)) *A class $\mathcal{K}$ of finite semigroups is a pseudovariety if and only if it consists of finite members of a generalised variety.* ❑

Next, we will consider the concepts of *existence varieties* and *bivarieties*. A class $\mathcal{K}$ of regular semigroups is said to be an *existence variety* or *e-variety* (Hall (1989)) if it is closed under taking $H(\mathcal{K})$, $S_r(\mathcal{K})$, and $P(\mathcal{K})$, where $S_r(\mathcal{K})$ denotes the class of all regular subsemigroups of semigroups in $\mathcal{K}$. When introducing the concept of existence varieties, T.E. Hall considered a regular semigroup as a unary semigroup by defining a map $'$: $S \rightarrow S$ which sends x to some fixed inverse x' (by the axiom of choice). By a family of *regular unary semigroup identities,* we mean a family of unary semigroup identities that includes the following pair of unary semigroup identities: $xx'x = x$ and $x'xx' = x'$. In such a case we refer to $'$ as an *inverse unary operation*. For any e-variety $\mathcal{V}$ of regular semigroups, we denote by $\mathcal{V}'=\{(S,.,'): (S,.)\in \mathcal{V}\}$ the set of all regular unary semigroups in $\mathcal{V}$; and by $Id(\mathcal{V}')$ we denote the set of all (unary semigroup) identities satisfied by all members of $\mathcal{V}'$. We denote by $\mathcal{RS}$ the e-variety of all regular semigroups.

Theorem 0:3.3 (Hall (1989)) *For any existence variety $\mathcal{V}$ of regular semigroups, and for any set of (unary semigroup) identities B containing $xx'x = x$ and $x'xx' = x'$, the following two conditions are equivalent:*

(i) B is a basis of $Id(\mathcal{V}')$;

(ii) $\mathcal{V}$ is given by

$\mathcal{V}= \{(S,.) \in \mathcal{RS}$: for some inverse unary operation $'$ on $(S,.)$, $(S,.,')$ satisfies $B\}$ and

$\mathcal{V} = \{(S,.) \in \mathcal{RS}$: for every inverse unary operation$'$ on $(S,.)$, $(S,.,')$ satisfies $B\}$.

❑

In the same year (1989) when the concept of *existence variety* first appeared in print, Kadourek and Szendrei published a paper on *bivarieties* of orthodox semigroups. It turns out that a bivariety of orthodox semigroup is just an existence variety consisting entirely of orthodox semigroups.

A *unary semigroup variety* is a class of semigroups closed under unary subsemigroups, arbitrary direct products, and homomorphic images. A *completely regular* semigroup is a disjoint union of groups; and, as defined earlier, an inverse semigroup is a regular semigroup with a unique inverse for each of its elements. Such a semigroup can be considered as a universal algebra with two operations: the binary operation of multiplication, and the unary operation of inversion $a \mapsto a^{-1}$: if S is completely regular, a is sent to the unique inverse of a in the maximal group containing a, otherwise, if S is inverse, a is sent to its unique inverse of a in S. We make the following observations:

(0:3.4) *The concepts of existence variety and unary semigroup variety coincide for the class of inverse semigroups and the class of completely regular semigroups* (see Jones and Trotter (1991(b)).

(0:3.5) *Every variety of semigroups consisting entirely of regular [orthodox] semigroups is an existence variety [bivariety]*

(0:3.6) *Every bivariety of orthodox semigroups is an existence variety*

(0:3.7) *Every variety of semigroups is a generalised variety* (see Ash (1985))

Let $\mathcal{F}$ be a class of identities, and $\mathcal{V}$ be the *equational class* determined by $\mathcal{F}$, in the sense that a semigroup S belongs to $\mathcal{V}$ if and only if it satisfies every identity in $\mathcal{F}$. In such a case we write $\mathcal{V} = [\mathcal{F}]$ and we call $\mathcal{V}$ a *variety* of semigroups. If $\mathcal{U}$ is a variety contained in $\mathcal{V}$ then $\mathcal{U}$ is a *subvariety* of $\mathcal{V}$. The next result is called the Birkhoff theorem.

Theorem 0:3.8 *Let $\mathcal{V}$ be a variety of algebras and $\mathcal{C}$ be a subclass of $\mathcal{V}$. Then $\mathcal{C}$ is a subvariety of $\mathcal{V}$ if and only if it is closed under arbitrary direct products, subalgebras, and homomorphic images.* ❑

We point out that the results presented in this section so far, and summarised in the table below, are in fact the Birkhoff-type theorems for each of the different systems of classification for semigroups, that we will be concerned with in the thesis. The word *variety* will be used to mean a semigroup variety. This is in contrast to *unary semigroup varieties,* often used in the literature to mean semigroup classes with the signatures $(\bullet,^{-1})$ as pointed out earlier (see Howie (1995))

Table: 0:3.9 Some systems of classifications for semigroups

System of classification	**Characterised as having closure properties under:**
Species	*Homomorphic images*
Existence varieties of regular semigroups	*Homomorphic images, regular subsemigroups, and arbitrary direct products*
Pseudovarieties of finite (monoids) semigroups	*Homomorphic images, finite direct products, and (submonoids) subsemigroups*
Generalised varieties of semigroups	*Homomorphic images, subsemigroups, arbitrary direct products, and arbitrary powers*
Varieties of semigroups	*Homomorphic images, subsemigroups, and arbitrary direct products*
Unary semigroup varieties	*Homomorphic images, unary subsemigroups, and arbitrary direct products*
Bivarieties of orthodox semigroups	*Homomorphic images, orthodox subsemigroups, and arbitrary direct products.*

0:4 LATTICES OF SEMIGROUP VARIETIES

As mentioned earlier, the set of all varieties forms a lattice. In this section, we will briefly summarise some of the sublattices that have been described:

A variety $\mathcal{V}$ of semigroups is said to be an *atom* if it has no proper subvariety other than the *trivial variety* $\mathcal{T} = [x = y]$. Kalicki and Scott described all such varieties precisely as follows (see Page 8 of Evans (1971)):

(0:4.1) $\mathcal{Z}_l = [xy = x]$, the variety of *left zero bands;*

(0:4.2) $\mathcal{N}_2 = [xy = yz]$, the variety of *null semigroups;*

(0:4.3) $\mathcal{Z}_r = [xy = y]$, the variety of *right zero bands*

(0:4.4) $\mathcal{S} = [xy = yx, x^2 = x]$, the variety of *semilattices;* and

(0:4.5) $\mathcal{A}_p = [x^p z = z, xy = yx]$, the variety of all *commutative groups of prime exponent p.*

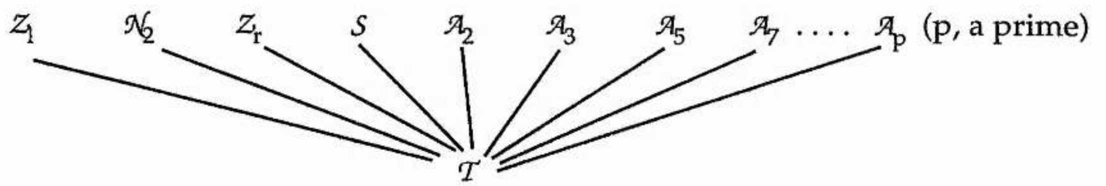


Figure 0:4.6 *The semigroup atoms*

As pointed out on Page 11 of Evans (1971), the complete join of all atoms forms the variety

(0:4.7) $\mathcal{D} = [zxyw = zyxw]$,

and the complete join of just the group atoms $\{\mathcal{A}_p$: where p is prime$\}$, produces the variety of all commutative semigroups

(0:4.8) $\mathcal{C} = [xy = yx]$.

Many attempts have been made to describe the lattice $\mathcal{L}(\mathcal{C})$ of all commutative semigroups, but have not been successful, and that lattice is still not fully known. It was shown by S. Burris and E. Nelson (see Evans (1971) or Volkov (1994)) that the lattice $\mathcal{L}(\mathcal{C})$ does not satisfy any non-trivial lattice laws; and thus the lattice structure of $\mathcal{L}(\mathcal{C})$ must be very complicated.

However, there is known to exist a large distributive sublattice of $\mathcal{L}(\mathcal{C})$, consisting of the following varieties:

(0:4.9) $$\mathcal{A} = \{\mathcal{A}_{(n,m)} : n \geq 0 \text{ and } m \geq 1\},$$

where each

$$\mathcal{A}_{(n,m)} = [z^n = z^{n+m}, xy = yx].$$

The varietal joins and meets involving members of this family can be expressed as follows (see Page 30 of Evans (1971)):

$$\mathcal{A}_{(k,l)} \vee \mathcal{A}_{(m,n)} = \mathcal{A}_{\max(k,m),\mathrm{lcm}(l,n)}$$

and

$$\mathcal{A}_{(k,l)} \wedge \mathcal{A}_{(m,n)} = \mathcal{A}_{\min(k,m),\gcd(l,n)}$$

An even larger family of varieties (not necessarily commutative), is referred to in the literature as the *Burnside varieties*:. These are:

(0:4.10) $$\mathcal{B}_{(m,n)} = [z^m = z^{m+n}].$$

In the case m = 1, namely the variety $\mathcal{B}_{(1,n)} = [z = z^{1+n}]$, consists entirely of regular semigroups. In particular, the case m = 1 = n gives the variety of all bands. The following will provide particularly interesting examples of varieties which consist entirely of groups:

(0:4.11) $$\mathcal{G}_m = [x^m y = y, yx^m = y]$$

(0:4.12) $$\mathcal{A}_m = [x^m y = y, yx = xy].$$

The lattice $\mathcal{L}(\mathcal{A}_m)$ is known to be isomorphic to the set of all positive divisors of m under division order; but the structure of $\mathcal{L}(\mathcal{G}_m)$ is not known (see Petrich (1974)).

The lattice $\mathcal{L}(\mathcal{B}_{(1,1)})$ of all varieties of bands — those semigroups satisfying the identity $x^2 = x$, was described completely and independently by Gerhard (1970), Burjukov (1970) and Fennemore (1970). Each variety of bands is determined within the equational class $[x^2 = x]$ by a single identity. In this thesis we will only be interested in the varieties of bands given in Figure 0:4.13 below.

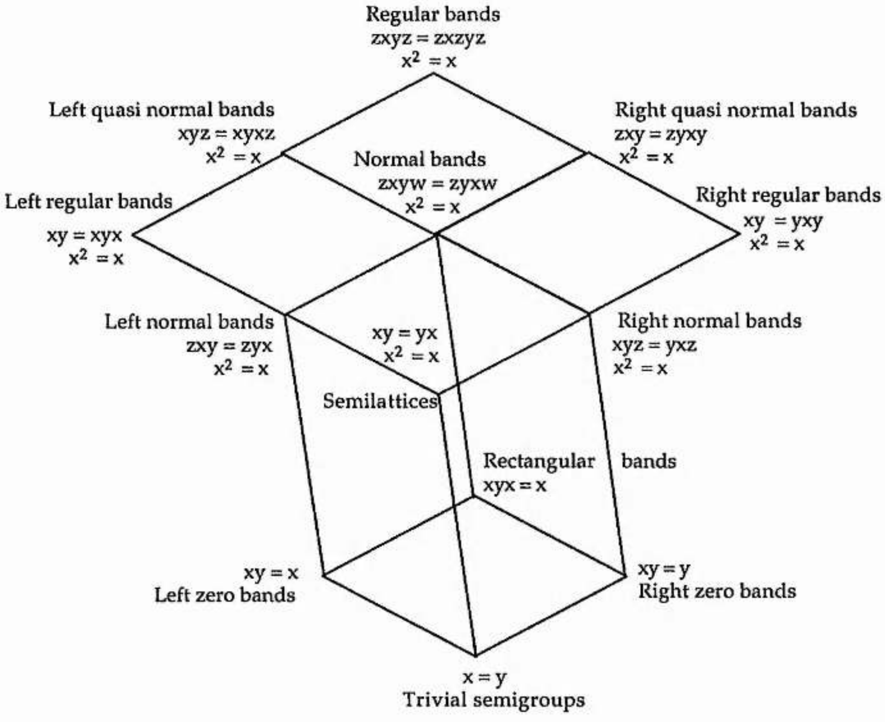


Figure 0:4.13 *The lattice of all varieties of regular bands*

The next family of commutative varieties was discovered by Reilly and Almeida (1984):

(0:4.14) $\mathcal{U} = \{\mathcal{U}_n : n = 1,2,3, \ldots\}$,

where each $\mathcal{U}_n$ is defined as follows:

$$\mathcal{U}_n = [x^2 = y^2, xy = yx, x_1x_2x_3 \ldots x_n = y_1y_2y_3 \ldots y_n] \quad = \quad [x^2 = y^2, xy = yx] \cap \mathcal{N}_n.$$

It was proved in Corollary 4.5 of that same paper, that the lattice $\mathcal{L}(\mathcal{U}_n)$ forms an ascending chain of length n-1:

i.e. $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_3 \subseteq \ \ldots . \subseteq \mathcal{U}_n$

The following lattice was first constructed by Melnik (1971), and later Petrich (1974) gave an alternative proof. We will have many occasion to refer to it.

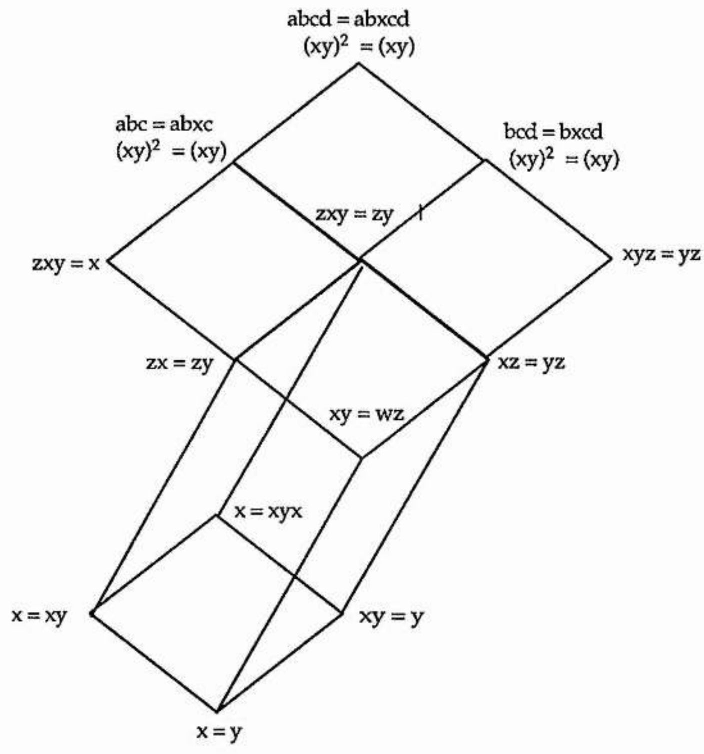


Figure 0:4.15 *The lattice* $\mathcal{L}(\mathcal{Z}_l \vee \mathcal{Z}_r)^2$ *of all varieties of 2-nilpotent extensions of rectangular bands.*

The semigroups involved in (0:4.14) form a subclass of a larger class of semigroups known as *nilpotent semigroups,* and nilpotent semigroups are those for which $S^n = \{0\}$ for some $n \geq 1$ (not necessarily commutative). A semigroup S is said to be an *n-nilpotent extension* of T if $S^n = T$ for some $n \geq 1$.

A semigroup S is said to be an *inflation* (Clifford and Preston (1962)) of a subsemigroup T if there exists a homomorphism ϕ from S onto T such that $S^2 \subseteq T$ and $x\phi = x$ for all $x \in S\phi$. The lattice below is formed by the subvarieties of the variety of all inflations of normal bands (see, Figure 1 of Evans (1971)). In Chapter 6 we describe the lattices formed by varieties consisting entirely of *n-inflations* of regular semigroups (see Page 51), a generalisation of the above concept due to Stojan Bogdanovich and Svetozar Milic (1987).

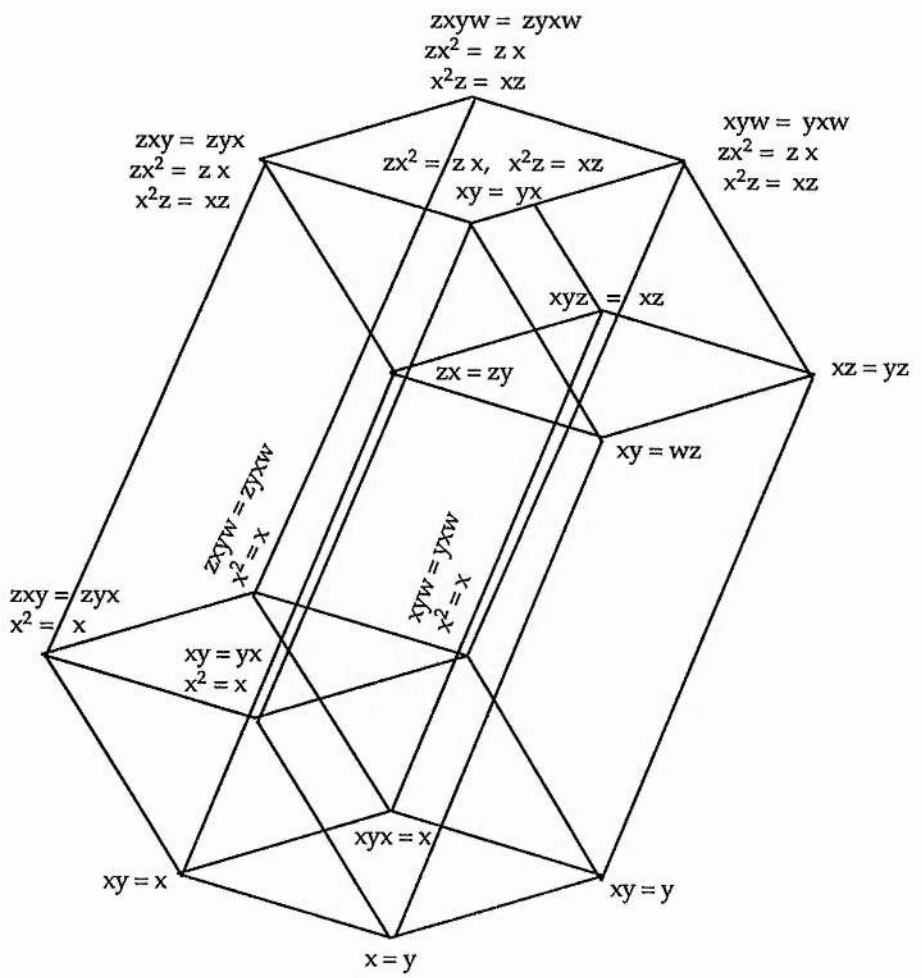


Figure 0:4.16 *The lattice of all varieties formed by inflations of normal bands*

Lemma 0:4.17 *If $\mathcal{V}$ is a variety of semigroups, then $\mathcal{V}^n$ = {S: $S^n \in \mathcal{V}$} is also a semigroup variety.*

Proof. We will show that $\mathcal{V}^n$ is closed under taking homomorphic images, arbitrary direct products, and subsemigroups. Take any semigroup S in $\mathcal{V}^n$, and any homomorphic image T = Sϕ. Then since $T^n = (S\phi)^n = S^n\phi \in \mathcal{V}$, T also belongs in $\mathcal{V}^n$. For any subsemigroup K of S, since $K^n \subseteq S^n$ and $\mathcal{V}$ is closed under subsemigroups, K also belongs to $\mathcal{V}^n$. Finally, let S be the direct product of an arbitrary family $\{S_\alpha : \alpha \in \Gamma\}$ of semigroups in $\mathcal{V}^n$. Now, we observe that

$$S^n = (\prod_{\alpha\in\Gamma} S_\alpha)^n = \{(a_\alpha)_{\alpha\in\Gamma} : a_\alpha \in S_\alpha{}^n\} = \prod_{\alpha\in\Gamma} S_\alpha^n$$

belongs to $\mathcal{V}$, proving that $\mathcal{V}^n$ is also closed under arbitrary direct products. Hence $\mathcal{V}^n$ forms a variety. ❑

The following is a paraphrased version of Theorem 6.6 of Petrich (1974). We will be able to generalise this result later (see Theorem 5:2.4).

Theorem 0:4.18 *For any subvarieties $\mathcal{V} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^2$, $\mathcal{Y} \subseteq \mathcal{G}_n$, and $\mathcal{X} \subseteq \mathcal{S}$, we have*

$$\mathcal{L}(\mathcal{V} \vee \mathcal{Y} \vee \mathcal{X}) \cong \mathcal{L}(\mathcal{V}) \times \mathcal{L}(\mathcal{Y}) \times \mathcal{L}(\mathcal{X}). \quad ❑$$

As a corollary of the above result, we have the following:

Corollary 0:4.19 *Let the varieties $(\mathcal{Z}_l \vee \mathcal{Z}_r)^2$, $\mathcal{S}$, $\mathcal{A}_n$, and $\mathcal{G}_n$ be as defined above. Then*

$$\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^2 \vee \mathcal{G}_n \vee \mathcal{S}) \cong \mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^2) \times \mathcal{L}(\mathcal{G}_n) \times \mathcal{L}(\mathcal{S})$$

and

$$\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^2 \vee \mathcal{A}_n \vee \mathcal{S}) \cong \mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^2) \times \mathcal{L}(\mathcal{A}_n) \times \mathcal{L}(\mathcal{S}). \quad ❑$$

The above result describes the lattice of all varieties formed by 2-nilpotent extensions of orthodox normal bands of groups, with maximal subgroups being derived from $\mathcal{G}_n$ or $\mathcal{A}_n$ respectively. The next two results will be useful latter. In the thesis we will denote by $\mathcal{L}(\Omega)$ the lattice of all varieties of semigroups.

Lemma 0:4.20 *Suppose that $\phi: \mathcal{L}(\mathcal{V}) \rightarrow \mathcal{L}(\Omega)$ is one to one, that it maps $\mathcal{L}(\mathcal{V})$ onto $\mathcal{L}(\mathcal{V}\phi)$ and that it preserves arbitrary meets. Then ϕ is an isomorphism from $(\mathcal{L}(\mathcal{V}),\wedge,\vee)$ onto $(\mathcal{L}(\mathcal{V}\phi),\wedge,\vee)$.*

Proof. Considered as lower semilattices, $(\mathcal{L}(\mathcal{V}),\wedge,\vee)$ and $(\mathcal{L}(\mathcal{V}\phi),\wedge,\vee)$ are isomorphic under ϕ. But since $(\mathcal{L}(\mathcal{V}),\wedge,\vee)$ is a lattice, and since ϕ is a bijection, it preserves the partial ordering on $(\mathcal{L}(\mathcal{V}),\wedge,\vee)$ and so $(\mathcal{L}(\mathcal{V}\phi),\wedge,\vee)$ is a lattice. But then $(\mathcal{L}(\mathcal{V}\phi),\wedge,\vee)$ is isomorphic to $(\mathcal{L}(\mathcal{V}),\wedge,\vee)$. ❑

Lemma 0:4.21 *Let* $\mathcal{V} = [p = q]$ *and* $\mathcal{V} \subseteq \mathcal{W} = [w = v]$. *If* $min\{|p|, |q|\} = n$, *then* $min\{|w|,|v|\} \geq n$.

Proof. Since $\mathcal{V} \subseteq \mathcal{W} = [w = v]$ and $\mathcal{V} = [p = q]$, the identity $w = v$ is a consequence of $p = q$. If $\min\{|w|,|v|\} < n$, then clearly $w = v$ can not be obtained as a consequence of $p = q$. By this contradiction, we must have $\min\{|w|,|v|\} \geq n$. ❑

A semigroup is *completely regular* if it is a pairwise disjoint union of a family of groups. Examples of such semigroups includes bands, groups, and *bands of groups*. A semigroup is a band of groups if there exists a congruence γ such that S/γ is a band and each γ-class is a group. The following result is a paraphrased version of Proposition 3 (ii) of Petrich (1975).

Proposition 0:4.22 *For any semigroup variety* $\mathcal{V}$ *consisting entirely of orthodox bands of groups,*

$$\mathcal{V} \subseteq [x = x^{n+1}], \text{ for some } n \geq 1. \qquad ❑$$

The next theorem is a paraphrased version of the main result and Lemma 1 of Petrich (1975).

Theorem 0:4.23 *For any semigroup variety* $\mathcal{V}$ *consisting entirely of orthodox bands of groups ,*

$$\mathcal{V} = (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G})$$

and

$$\mathcal{L}(\mathcal{V}) \cong \mathcal{L}(\mathcal{V} \cap \mathcal{B}) \times \mathcal{L}(\mathcal{V} \cap \mathcal{G}), \tag{0:4.24}$$

where $\mathcal{B}$ *and* $\mathcal{G}$ *denote the classes of all bands and groups, respectively.* ❑

A large portion of this thesis involves describing the lattices of semigroup varieties formed by semigroups consisting entirely of some special types of nilpotent extensions of regular semigroups. For a brief review of work previously done in this direction, we draw the reader's attention to the following references: Melnik (1971), Petrich ((1974), Gerhard (1977(a) and 1977(b)), Volkov and Ershova (1990). In fact, in 1971 Melnik published his construction of the lattice formed by all varieties of semigroups whose square is a rectangular band. Petrich in 1974 described the lattice of all varieties consisting of semigroups S for which S^2 is an orthodox normal band of groups. In Gerhard's paper labelled 1977(a) and 1977(b), the lattice of all varieties of semigroups with idempotent square (i.e. varieties satisfying $(xy)^2 = xy$) was investigated. In particular, he proved this lattice to be distributive and he even provided a subdirect decomposition for the lattice. In 1990, Volkov and Ershova proved, among other things, that the lattice of all semigroup varieties satisfying the identity $(xy)^k = xy$, $k \geq 1$, is modular (and even arguesian), analogous to some earlier results proved for completely regular semigroups by Pastijn (1990). In fact, it was Polak who in (1985) and (1987), provided the essential understanding of the lattice formed by varieties of completely regular semigroups.

In Chapters 4 - 7 of this thesis, we describe lattice structures of semigroup varieties formed entirely by nilpotent extensions of some (special) classes of regular semigroups. In fact we generalise Petrich's 1974 results in Chapters 4 and 5, where we consider certain varieties formed by semigroups S for which S^n is an orthodox normal band of groups, $n \geq 1$. Some further progress is made in Chapter 6, in the direction taken by Volkov and Ershova, where we describe certain lattices formed by semigroups which are n-inflations of regular semigroups, and this includes certain varieties of semigroups which are determined by identities of the type:

$$(x_1x_2x_3 \ldots x_t)^n = x_1x_2x_3 \ldots x_t, \qquad \text{for some } n \geq 1 \text{ and } t \geq 1.$$

The thesis ends in Chapter 7 with a description of the skeleton of the lattice formed by varieties consisting entirely of nilpotent extensions of rectangular bands.

CHAPTER 1

A FAMILY OF SEMIGROUP CONGRUENCES

We introduce a countable directed family of semigroup congruences in this chapter, and a theory analogous to the theory of normal series for groups is developed. This interesting family of semigroup congruences, surprisingly, turns out to be an effective tool for studying the lattice-structures of the lattices formed by certain species of semigroups (classes of semigroups closed under taking homomorphic images) such as varieties, pseudovarieties, and existence varieties etc.

1:1 HOMO-SERIES OF SEMIGROUPS

A *homo-series* $\{H_i\}$ of a semigroup S is a family of quotients: $\{S/\omega : \omega \in Q\}$ where Q is a directed family of congruences on S, assumed to contain the identity relation. The ordering in Q is the usual containment relation of congruences. Note that if $\omega \supseteq \gamma$ then S/ω is a homomorphic image of S/γ, and we write $S/\gamma \rightarrow S/\omega$. Any two isomorphic semigroups are regarded as being equal, and each homo-series of a semigroup S contains S since Q contains the identity relation.

We say a homo-series$\{K_i\}$ is a *refinement* of the homo-series $\{H_j\}$ if $\{H_j\} \subseteq \{K_i\}$, that is if every member of $\{H_j\}$ is a member of $\{K_i\}$. Moreover, a homo-series $\{K_i\}$ is said to be a *homomorphic image* of another homo-series $\{H_j\}$ if there exists a mapping ϕ from $\{H_j\}$ onto $\{K_i\}$ such that the semigroup $X\phi$ is a homomorphic image of X for every X in $\{H_j\}$. In particular, if ϕ is one to one and if $X\phi$ is isomorphic to X for every X, then we will say that these homo-series are *isomorphic*.

For each ordered pair (n,m) of non-negative integers, consider the relation $\theta^S(n,m)$ on a semigroup S given by:

(1:1.1) $\theta^S(n,m) = \{(a,b)\text{: } uav = ubv, \text{ for all } u \in S^n \text{ and } v \in S^m\},$

where S^n is the set of all elements of S which are products of n-elements, and we make the convention that $S^1 = S$ and S^0 is the set containing only the empty word. It is clear that $\theta(n,m)$ is an equivalence relation on S. We will show that it is a congruence. Take any $(a,b) \in \theta(n,m)$ and any element $s \in S$. Then for all $u \in S^n$ and $v \in S^m$, since $S^{n+1} \subseteq S^n$ and $S^{m+1} \subseteq S^m$, we have that:

$$u(as)v = u(a)(sv) = u(b)(sv) = u(bs)v$$

and

$$u(sa)v = (us)av = (us)bv = u(sb)v\ ;$$

and so by the first set of equalities we have $(as,bs) \in \theta(n,m)$, and from the second set we have $(sa,sb) \in \theta(n,m)$. Thus we have shown that $\theta(n,m)$ is indeed a congruence on S. Whenever there is no ambiguity as to which semigroup S we are talking about, we shall write $\theta^S(n,m)$ simply as $\theta(n,m)$. In the literature, a semigroup S is called *left reductive* if $\theta(1,0)$ is the identity relation on S, and is *right reductive* if $\theta(0,1)$ is the identity relation. A semigroup that is both left and right reductive is called *reductive* . In Clifford and Preston (1961), the particular case $\theta(1,0)$ is called the *kernel of the right regular representation* of S.

The following particular homo-series will be referred to as the *normal series of S*:

$$\{N_{i,j}\} = \{S/\theta^S(i,j):\ 0 \le i \le n \text{ and } 0 \le j \le m\}.$$

Lemma 1:1.2 *For any semigroup S, both the congruences given below are equal to the congruence* $\theta^S(1,1)$:

$$\sigma\,(l,r) = \{(a,b): (ax,bx) \in \theta^S(1,0) \text{ for all } x \in S\}$$

and

$$\sigma(r,l) = \{(a,b): (za,zb) \in \theta^S(0,1) \text{ for all } z \in S\}.$$

Hence , for every $(n,m), (k,i) \in \mathbf{N}^{[0]} \times \mathbf{N}^{[0]}$

$$T/\theta^T(k,i) \;\cong\; S/\theta^S(n+k,m+i) \;\cong\; R/\theta^R(n,m);$$

where $T = S/\theta^S(n,m)$ *and* $R = S/\theta^S(k,i)$.

Proof. Now,

$$\begin{aligned}\sigma(r,l) &= \{(a,b): (ax,bx) \in \theta^S(1,0) \text{ for all } x \in S\}\\ &= \{(a,b): zax = zbx \text{ for all } x,z \in S\}\\ &= \theta^S(1,1)\\ &= \{(a,b): (za,zb) \in \theta^S(0,1) \text{ for all } z \in S\}\\ &= \sigma(l,r).\end{aligned}$$

It follows that

$$S/\theta(1,1) = S/\sigma(r,l) = S/\sigma(l,r),$$

Now, let $T = S/\theta^S(1,0)$ and $R = S/\theta^S(0,1)$ so that $(n,m) = (1,0)$ and $(k,i) = (0,1)$. We observe that

$$\sigma(r,l) = \mathit{ker}\,(\theta^S(1,0)^{\natural} \circ \theta^T(0,1)^{\natural})$$

and

$$\sigma(l,r) = \mathit{ker}\,(\theta^S(0,1)^{\natural} \circ \theta^R(1,0)^{\natural}).$$

Therefore,

$$S/\sigma(r,l) \cong T/\theta^T(0,1) \qquad \text{and} \qquad S/\sigma(l,r) \cong R/\theta^R(1,0)$$

under the isomorphisms:

$$a\sigma(r,l) \mapsto a\mathit{ker}\,(\theta^S(1,0)^{\natural} \circ \theta^T(0,1)^{\natural}), \quad \text{for each } a \in S$$

and

$$a\sigma(l,r) \mapsto a\mathit{ker}\,(\theta^S(0,1)^{\natural} \circ \theta^R(1,0)^{\natural}), \quad \text{for each } a \in S.$$

Consequently,

$$R/\theta^R(1,0) \;\cong\; S/\theta^S(1,1) \cong\; T/\theta^T(0,1);$$

and the following diagram commutes:

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & \\
S/\theta(1,0) \arrow[dr] & & S/\theta(0,1) \arrow[dl] \\
 & S/\theta(1,1) &
\end{tikzcd}$$

To prove the general case, let $T = S/\theta(n,m)$ and consider the congruence:

$\mathit{ker}\,(\theta^S(n,m)^{\natural} \circ \theta^T(k,i)^{\natural})$

$= \{(a,b) : a\theta^S(n,m)^{\natural} \circ \theta^T(k,i)^{\natural} = b\theta^S(n,m)^{\natural} \circ \theta^T(k,i)^{\natural}\}$

$= \{(a,b) : (a\theta^S(n,m)\,,\, b\theta^S(n,m)) \in \theta^T(k,i)\}$

$= \{(a,b) : (uav)\theta^S(n,m) = (ubv)\theta^S(n,m)$ for all $u\theta^S(n,m) \in T^k$ and

$v\theta^S(n,m) \in T^i\}$

$= \{(a,b) : (uav)\theta^S(n,m) = (ubv)\theta^S(n,m)$ for all $u \in S^k$ and $v \in S^i\}$

$= \{(a,b) : uav = ubv$ for all $u \in S^k$ and $v \in S^i\}$

$= \theta^S(n+k,m+i);$

and thus we have shown that

$$S/\theta(n+k,m+i) = S/\mathit{ker}(\theta^S(n,m)^{\natural} \circ \theta^T(k,i)^{\natural}),$$

and the quotients $S/\theta^S(n+k,m+i)$ and $T/\theta^T(k,i)$ are isomorphic under the map:

$$a\theta(n+k,m+i) \mapsto a\,ker(\theta^S(n,m) \circ \theta^T(k,i)),$$

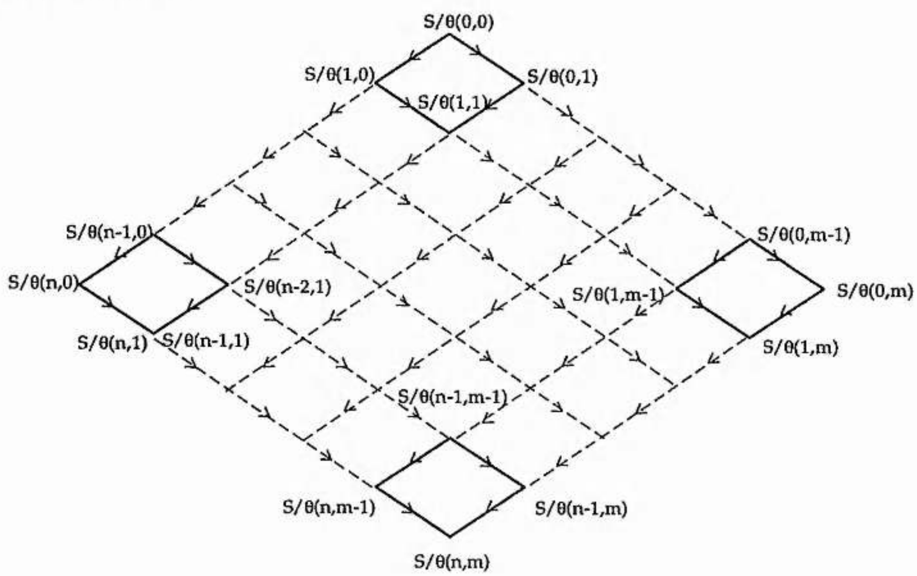
for each a in S. Similarly, it can be shown that

$$S/\theta^S(n+k,m+i) \;\cong\; R/\theta^R(n,m),$$

where $R = S/\theta(k,i)$. ❑

Define $(n,m) \leq (s,t)$ if and only if both $n \leq s$ and $m \leq t$ on the set of all non negative integers. For any semigroup S, call the least ordered pair (n,m), if any exists (see comments after Corollary 1:1.6), such that $S/\theta(n,m)$ is [left,right] reductive as the *[left,right] reductive index* of S, and call S an *(n,m)-[left,right] reductive semigroup.* We point out also that for any (n,m)-reductive semigroup, and any $(n,m) \leq (s,t)$ the congruence $\theta(n,m) = \theta(s,t)$. As shown in the last part of Example 1:1.7, there is a semigroup without a reductive index.

Corollary 1:1.3 *For any semigroup S and any $(n,m) \geq (0,0)$, the following diagram commutes:*



where the homomorphisms are :

$$S/\theta^S(i,j) \longrightarrow S/\theta^S(i,j+1), \textit{ is } \; a\theta^S(i,j) \mapsto a\theta^S(i,j+1), \textit{for each } a \in S$$

and

$$S/\theta^S(i,j) \longrightarrow S/\theta^S(i+1,j), \textit{ is } \; a\theta^S(i,j) \mapsto a\theta^S(i+1,j), \textit{for each } a \in S$$

and any ordered pair (i,j) *of non-negative integers.*

Proof. These maps are well defined onto homomorphisms since the containments $\theta^S(i,j) \subseteq \theta^S(i+1,j)$ and $\theta^S(i,j) \subseteq \theta^S(i+1,j)$ hold. ❑

Theorem 1:1.4 *For any semigroup* S, *and any homomorphism* ψ *from* S *onto* T, *there exists a family of onto homomorphisms :*

$$\psi_{i,j}: \ S/\theta^{S}(i,j) \rightarrow T/\theta^{T}(i,j)\,,$$

defined by

$$[a\theta^{S}(i,j)]\psi_{i,j} = (a\psi)\theta^{T}(i,j) \qquad (a \in S).$$

for $i = 0,1,2,3,4\,, \ldots, n$ *and* $j = 1,2,3,\ldots, m$, *such that the diagram below commutes:*

$$\begin{tikzcd}
S \arrow[r, "\psi"] \arrow[d, "\theta^{S}(i,j)"'] & T \arrow[d, "\theta^{T}(i,j)"] \\
S/\theta^{S}(i,j) \arrow[r, "\psi_{i,j}"] & T/\theta^{T}(i,j)
\end{tikzcd}$$

Proof. Since the map ψ is a homomorphism from S onto T, we have that for any $(i,j) \le (n,m)$,

$$(S^{i})\psi = (S\psi)^{i} = T^{i} \qquad \text{and} \qquad (S^{j})\psi = (S\psi)^{j} = T^{j},$$

and for any $u \in T^{i}$ and $v \in T^{j}$ there exist $x \in S^{i}$ and $y \in S^{j}$ that $u = x\psi$ and $v = y\psi$. Then, for any $(a,b) \in \theta^{S}(i,j)$ and for all $u \in T^{i}$ and $v \in T^{j}$, we have

$$u(a\psi)v = (x\psi)(a\psi)(y\psi) = (xay)\psi = (xby)\psi = (x\psi)(b\psi)(y\psi) = u(b\psi)v.$$

Hence $(a\psi, b\psi) \in \theta^{T}(i,j)$, and thus the map ψ_{ij} is well defined. We see that ψ_{ij} is in fact a homomorphism, since for any a, b $\in$ S we have

$$\begin{aligned}
[a\theta^{S}(i,j)\psi_{i,j}\,]\,[b\theta^{S}(i,j)\psi_{i,j}\,] &= [a\psi\theta^{T}(i,j)]\,[b\psi\theta^{T}(i,j)] \\
&= (ab)\psi\theta^{T}(i,j) \\
&= [(ab)\theta^{S}(i,j)]\psi_{i,j}\,. \qquad \square
\end{aligned}$$

The above theorem implies that for any homo-series $\{H_t\}$ of S there exists a refinement, namely

(1:1.5) $$\{K_r\} = \{T/\theta^{T}(n,m) : T \in \{H_t\} \text{ and for every } (n,m)\},$$

which turns out also to be a refinement of the normal series $\{N_{i,j}\}$. This is so because every homo-series of S contains S. Moreover, it also implies that two semigroups are homomorphic [isomorphic] if and only if their normal series are homomorphic [isomorphic].

Every semigroup has at least one reductive congruence, namely the universal relation $\{(a,b): a,b \in S\}$. We claim that every semigroup has a least reductive congruence, but is not always of the type $\theta(n,m)$. To show that there exists a least reductive congruence, take any reductive congruences δ,ω on S, and any elements a and b of S such that $(xa,xb) \in \delta\cap\omega$ for all x in S. Then this implies that $(xa,xb) \in \delta$ and $(xa,xb) \in \omega$ for all x in S. By the left reductivity of S/δ and S/ω, we have that $(a,b) \in \delta$ and $(a,b) \in \omega$. This proves that $(a,b) \in \delta\cap\omega$, and so $\delta\cap\omega$ is a left reductive congruence. Similarly, one can show that $\delta\cap\omega$ is also right reductive. Hence $\delta\cap\omega$ is a reductive congruence on S. It follows that S has a least reductive congruence, namely the intersection of all its reductive congruences.

In general, the congruences of the type $\theta(n,m)$ are not all reductive. However, the following result shows that if $\theta(n,m)$ is a reductive congruence, then it turns out to be the least reductive congruence on that particular semigroup.

Corollary 1:1.6 *For any semigroup S if the congruence $\theta(n,m)$ is reductive, then it is the least reductive congruence.*

Proof. Suppose that $\theta(n,m)$ is a reductive congruence, and take any reductive congruence γ on S. Then from Theorem 1:1.4 there exists a congruence ω on $S/\theta(n,m)$ such that

$$\gamma \circ \theta^T(n,m) = \theta^S(n,m) \circ \omega, \text{ where } T = S/\gamma$$

But since $T = S/\gamma$ is reductive, we have that $\theta^T(n,m)$ is the identity relation on T; and the conclusion follows immediately. ❑

By saying that $S/\theta^S(n,m)$ is *left reductive,* we really mean that for any elements a and b of S, if $(xa)\theta^S(n,m) = (xb)\theta^S(n,m)$ for all x in S, then $(a,b) \in \theta^S(n,m)$. Dually, $S/\theta^S(n,m)$ is *right reductive,* if $(ax)\theta^S(n,m) = (bx)\theta^S(n,m)$ for all x in S implies that $(a,b) \in \theta^S(n,m)$. We define $S/\theta^S(n,m)$ to be *reductive* if it is both left and right reductive.

We will show that if $S/\theta^S(n,m)$ is reductive, then for every $(r,k) \geq (n,m)$ the quotient $S/\theta^S(r,k)$ is also reductive. For any such $(r,k) \geq (n,m)$, there exists $(u,v) \geq (0,0)$ such that

$$(r,k) = (u,v) + (n,m).$$

Since $T = S/\theta^S(n,m)$ is reductive, and $T/\theta^T(u,v) \cong S/\theta^S(r,k)$ by Corollary 1:1.2, the congruence $\theta^T(u,v)$ is the identity relation on T. It follows that the equality $\theta^S(n,m) = \theta^S(r,k)$ holds and so $S/\theta^S(r,k)$ is reductive. This proves that $\theta^S(r,k)$ is also a reductive congruence.

In particular, suppose that S is a reductive semigroup. Then by definition, both $\theta^S(1,0)$ and $\theta^S(0,1)$ reduce to the identity relation and so:

$$S/\theta^S(1,0) = S/\theta^S(0,0) = S/\theta^S(0,1) = S.$$

From what we have proved earlier, for every ordered pair $(i,j) \geq (0,0)$ of non-negative integers, $S/\theta^S(i,j) = S$ and that the congruence $\theta^S(i,j)$ reduces to the identity relation on S.

Suppose for a semigroup S, that both $\theta^S(n,m)$ and $\theta^S(r,k)$ are reductive congruences. Then from Corollary 1:1.6, $\theta^S(n,m) = \theta^S(r,k)$ is the least reductive congruence. However, it is still not known as to whether or not there exists a least ordered pair (u,v) of non-negative integers such that $\theta^S(u,v)$ is the least reductive congruence.

Take any two distinct ordered pairs (n,m) and (r,k) of non-negative integers such that the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\theta^S(r,k)"] \arrow[d, "\theta^S(n,m)"'] & T = S/\theta^S(r,k) \arrow[d, "\theta^T(n,m)"] \\
S/\theta^S(n,m) \arrow[r, "f"'] & S/\theta^S(l,t)
\end{tikzcd}$$

where $f : S/\theta^S(n,m) \longrightarrow S/\theta^S(l,t)$ is defined by $a\theta^S(n,m) \mapsto a\theta^S(l,t)$ for each $a \in S$. Then this map is a well defined onto homomorphism if and only if the containment

$$\theta^S(n,m) \subseteq \theta^S(l,t)$$

holds. This inclusion is possible in the following cases:

(a) If both $\theta^S(n,m)$ and $\theta^S(l,t)$ are reductive congruences on S, then from Corollary 1:1.6, $\theta^S(n,m) = \theta^S(l,t)$ is the least reductive congruence on S and so f is a well defined onto homomorphism.

(b): If $(n,m) \leq (l,t)$ then clearly $\theta^S(n,m) \subseteq \theta^S(l,t)$ and so the map f is a well defined homomorphism.

Example 1:1.7 Let $2 \leq p$, $1 \leq r$ and consider the monogenic semigroup

$$K(p,p+r)= \{a^1, a^2 \ldots, a^{p-1}, a^p, a^{p+1}, \ldots, a^{p+r-1}\}, \text{ with } a^{p+r} = a^p.$$

It can be shown that the $\theta(1,0)$-classes of this semigroup are precisely:

$$\{a^{p-1}, a^{p+r-1}\}, \{a^1\}, \{a^2\}, \{a^3\}, \ldots, \{a^{p-2}\}, \{a^p\}, \{a^{p+1}\}, \{a^{p+2}\}, \ldots, \{a^{p+r-2}\}.$$

Note that the set $\{a^{p-1}, a^{p+r-1}\}$ forms the only non-singleton $\theta(1,0)$-class. It is an easy exercise to verify that

$$S/\theta(1,0) \cong K(p-1, p+r-1),$$

where $S = K(p,p+r)$. By repeatedly taking such congruences on the quotients so obtained, we have the following homo-series:

$$K(p,p+r) \to K(p-1,p+r-1) \to K(p-2, p+r-2) \to \ldots \to K(2,r+2) \to K(1,r+1),$$

which terminates at $K(1,r+1)$ since it is a group. Since $K(p,p+r)$ is commutative, its normal series forms a chain of length p-1. Now, to show that there are semigroups with normal series of infinite length, fix a positive integer r and consider the direct product P of the family of monogenic semigroups:

$$\{K(p,p+r): p = 1,2,3,4, \ldots \}.$$

It can be shown that for every p and any pair of non-negative integers i and j, such that $i+j \leq p$, the quotient $K(p,p+r)/\theta(i,j)$ is not reductive. Hence it follows from Lemma 1:1.9 that for every pair (i,j), the quotient $P/\theta(i,j)$ is not reductive. ❑

Example 1:1.8 For regular semigroups, and more generally for semigroups S such that $S^2 = S$, the congruence $\theta(1,1)$ is reductive, while $S/\theta(1,0)$ is left reductive; hence the normal series of S can form at most a four-element diamond. In particular, if S is a generalised inverse semigroup [a normal band] then one can show that: $S/\theta(1,0)$ is right generalised inverse [a right normal band]; $S/\theta(0,1)$ is left generalised inverse [a left normal band]; and $S/\theta(1,1)$ is an inverse semigroup [a semilattice]. Since inverse semigroups [semilattices] are reductive, the normal series terminates at $S/\theta(1,1)$ (see Corollary 3:5.19). ❑

Define a *direct product* $\{H_i\} \times \{K_j\}$ of two homo-series $\{H_i\}$ and $\{K_j\}$ as the set of all direct products of the semigroups in these homo-series.

$$\text{i.e.} \qquad \{H_i\} \times \{K_j\} = \{ H \times K: H \in \{H_i\}, K \in \{K_j\}\}.$$

Lemma 1:1.9 *Let* $S = \prod_{\alpha \in \Gamma} S_\alpha$ *be the direct product of an arbitrary family* $\{S_\alpha : \alpha \in \Gamma\}$ *of semigroups. Then for any ordered pair (n,m) of non negative integers,*

$$S / \theta^S(n,m) \cong \prod_{\alpha \in \Gamma} (S_\alpha / \theta^{S_\alpha}(n,m));$$

and each $\theta^S(n,m)$*-class is a Cartesian product of some* $\theta(n,m)$*-classes of the semigroups in* $\{S_\alpha : \alpha \in \Gamma\}$*. Hence, the normal series of S is isomorphic to the direct product of the normal series of the semigroups* $\{S_\alpha : \alpha \in \Gamma\}$.

Proof. Take any elements $a = (a_\alpha)_{\alpha \in \Gamma}$ and $b = (b_\alpha)_{\alpha \in \Gamma}$ of S. We observe that $(a,b) \in \theta^S(n,m)$ if and only if $(a_\alpha, b_\alpha) \in \theta^{S_\alpha}(n,m)$ for each $\alpha \in \Gamma$. The conclusions follow easily. ❑

The free monogenic semigroup is reductive since it is isomorphic to the set of all positive integers under addition and has the strange property of cancellativity. However, the free monogenic 3-nilpotent semigroup is isomorphic to the monogenic semigroup K(3,4) = $\{a^i : a^3 = a^4, \text{ for } i \geq 1\}$, which is not reductive. Therefore the property of reductivity is not closed under homomorphic images. It is clear that since every semigroup can be embedded in a monoid, this property is also not closed under subsemigroups. However, we have, by Lemma 1:1.9, that reductivity is closed under taking arbitrary direct products.

1:2 SEMIGROUP DIGRAPHS

A *digraph* $G = (V_G, A_G)$ consists of a non-empty set V_G referred to as the *vertices* of G, and a family A_G of *arrows* connecting pairs of vertices (and arrow directions do matter). For any $(\alpha,\beta) \in V_G \times V_G$, if we denote by $arr(\alpha,\beta)$ the set of all arrows from α to β, then

$$A_G = \cup\{ arr(\alpha,\beta) : (\alpha,\beta) \in V_G \times V_G\}.$$

Note that $arr(\alpha,\beta)$ may be empty, or may contain more than one arrow.

A *set-digraph* is a digraph $G = (V_G, A_G)$, where the vertices are (pairwise disjoint) non-empty sets, say $V_G = \{S_\alpha : \alpha \in \Gamma\}$, and a collection of maps between these sets form the arrows A_G of the digraph G. A vertex in this case is a set, rather than a single object as in the usual sense of digraphs.

A set-digraph $G = (V_G, A_G)$ will be called a *left semigroup digraph* if the three additional conditions given below are satisfied:

(1:2.1) The indexing set Γ of $V_G = \{S_\alpha : \alpha \in \Gamma\}$ is a semigroup.

(1:2.2) For each $(\alpha,\beta) \in \Gamma \times \Gamma$ there exists a map, say

$$\Phi_{\alpha,\beta} : S_\alpha \longrightarrow S_{\alpha\beta},$$

and so $\Phi_{\alpha,\beta} \in arr(S_\alpha, S_{\alpha\beta}) \neq \varnothing$. Moreover,

$$arr(S_\alpha, S_{\alpha\beta}) = \{\Phi_{\alpha,\gamma} : \alpha\beta = \alpha\gamma,\ \gamma \in \Gamma\}.$$

(1:2.3) For any α,β,γ in Γ we have the following transitivity relation

$$\Phi_{\alpha,\beta}\Phi_{\alpha\beta,\gamma} = \Phi_{\alpha,\beta\gamma}.$$

Given any family of sets $\{S_\alpha : \alpha \in \Gamma\}$, say, together with a collection of maps between these sets – forming a semigroup digraph, we now plan to show that an associative binary operation can be defined on the union $S = \cup\{S_\alpha : \alpha \in \Gamma\}$ making S a semigroup. Conversely, we will show that every semigroup may be constructed this way.

Now, take any semigroup S and denote by $\nabla(S) = \{S_\alpha : \alpha \in \Gamma\}$ the family of all $\theta(1,0)$-classes of S, and $\Gamma = S/\theta(1,0)$. For each $(\alpha,\beta) \in \Gamma \times \Gamma$ define a map:

(1:2.4) $\phi_{\alpha,\beta} : S_\alpha \longrightarrow S_{\alpha\beta}$, $x \mapsto xb$, for any $b \in S_\beta$;

since any two elements b and b' of S_β are $\theta(1,0)$-related, we have $xb' = xb$ and so this map is well defined. Denote by M(S) the family of all such maps between the $\theta(1,0)$-classes of S:

$$M(S) = \{\phi_{\alpha,\beta}\colon S_\alpha \longrightarrow S_{\alpha\beta}\,,\ (\alpha,\beta) \in \Gamma\times\Gamma\}.$$

We remark that in Kopamu (1991) and (1994), the map $\phi_{\alpha,\beta}$ is denoted by $\phi^{l}_{\alpha,\alpha,\beta}$ and is referred to as an *l-structure map*.

Lemma 1:2.5 *For any semigroup* S, *the pair* $G = (\nabla(S), M(S))$ *forms a left semigroup digraph.*

Proof. The quotient $\Gamma = S/\theta(1,0)$ is a semigroup, and it indexes the family $\nabla(S)$ of all $\theta(1,0)$-classes of S. Moreover, from the way in which each map in M(S) is defined in (1:2.4), they satisfy (1:2.2). We need only show that the members of M(S) satisfy the transitivity condition of (1:2.3). Take any $\alpha,\beta,\gamma \in \Gamma$ and any $a \in S_\alpha$, $b \in S_\beta$, and $c \in S_\gamma$.
Then

$$(a\phi_{\alpha,\beta})\phi_{\alpha\beta,\gamma} = (ab)\phi_{\alpha\beta,\gamma} = (ab)c = a(bc) = a\phi_{\alpha,\beta\gamma}\,.$$

We have the last equality since bc is contained in $S_{\beta\gamma}$. ❑

The following result first appeared in Kopamu (1991). We present here a graph theoretic statement, and a proof is included for the sake of completeness. It enables us to construct interesting examples.

Theorem 1:2.6 *Let* (V_G, A_G) *be a left-semigroup digraph, where* V_G *is a family of pairwise disjoint non-empty sets, say* $V_G = \{S_\alpha\colon \alpha \in \Gamma\}$.
Define a binary operation on $S = \bigcup\{S_\alpha\colon \alpha \in \Gamma\}$ *as follows: for any a in* S_α, *and b in* S_β, $a \blacklozenge b = a\phi_{\alpha,\beta}$.
Then $(S, \blacklozenge)$ *is a semigroup.*

Conversely, for every semigroup S there exists a left semigroup digraph, namely $G = (\nabla(S), M(S))$, *such that S is constructed in the way described above.*

Proof. Take any left semigroup digraph $G = (V_G, A_G)$, where $V_G = \{S_\alpha\colon \alpha \in \Gamma\}$. Then for any $\alpha,\beta,\gamma \in \Gamma$, and any elements $a \in S_\alpha$, $b \in S_\beta$, and $c \in S_\gamma$ we have that

$$\begin{aligned}(a\blacklozenge b)\blacklozenge c = (a\phi_{\alpha,\beta})\blacklozenge c &= a\phi_{\alpha,\beta}\phi_{\alpha\beta,\gamma}\\ &= a\phi_{\alpha,\beta\gamma} && \text{(by the transitivity condition (1:2.3))}\\ &= a\blacklozenge(b\phi_{\beta,\gamma}) && (\text{since } b\phi_{\beta,\gamma} \in S_{\beta\gamma})\\ &= a\blacklozenge(b\blacklozenge c)\,.\end{aligned}$$

The converse follows immediately from Lemma 1:2.5 and the preceding discussions. ❑

The above result shows that semigroup digraphs do play an important role in this structure theory. We also point out that the well known concept of *strong semilattices* of semigroups (see Howie (1976)), formed by families of pairwise disjoint semigroups, can also be considered as semigroup digraphs, but of course the multiplication there is defined differently.

We now complete this section by describing a way of producing concrete examples of semigroups constructed in this way. More precisely, we show that for each semigroup Γ there exist at least one semigroup S, different from Γ, such that $S/\theta(1,0) = \Gamma$.

Construction 1:2.7 Take any semigroup Γ and embed it into the semigroup $\mathcal{T}(\Gamma^{(1)})$ of all full transformations of $\Gamma^{(1)}$ under the right regular representation of Γ (see Clifford and Preston (1962)) as follows:

$$\rho : \Gamma \longrightarrow \mathcal{T}(\Gamma^{(1)}), \quad \alpha \mapsto \rho_\alpha,$$

where $\rho_\alpha : \Gamma^{(1)} \longrightarrow \Gamma^{(1)}$ is the map defined by $x \mapsto x\alpha$ for each $x \in \Gamma^{(1)}$. Now take a family of pairwise disjoint sets, say $\{S_\alpha : \alpha \in \Gamma\}$, such that $|S_\alpha| = |\Gamma^{(1)}|$ for each $\alpha \in \Gamma$; and fix any bijection g_α from $\Gamma^{(1)}$, say

$$g_\alpha : \Gamma^{(1)} \longrightarrow S_\alpha .$$

Then for each $(\alpha,\beta) \in \Gamma \times \Gamma$, define a map from S_α into $S_{\alpha\beta}$ as follows:

(1:2.8) $\qquad \phi_{\alpha,\beta}: S_\alpha \longrightarrow S_{\alpha\beta}, \qquad x \mapsto (x)g_\alpha^{-1}\rho_\beta g_{\alpha\beta} .$

Since the map g_α is a bijection, g_α^{-1} exists and so the map (1:2.8) is well defined. On the set $S = \bigcup\{S_\alpha : \alpha \in \Gamma\}$ define a binary operation $\blacklozenge$ by the rule that for any $a \in S_\alpha$, $b \in S_\beta$, $a \blacklozenge b = a\phi_{\alpha,\beta}$. Then to show that $(S,\blacklozenge)$ is a semigroup, we need only show that the transitivity law of (1:2.3) is satisfied by this family of maps. For any $\alpha \in \Gamma$ and any element $a \in S_\alpha$,

$$(a\,\phi_{\alpha,\beta})\,\phi_{\alpha\beta,\gamma} = (ag_\alpha^{-1}\,\rho_\beta g_{\alpha\beta})g_{\alpha\beta}^{-1}\,\rho_\gamma g_{\alpha\beta\gamma} = ag_\alpha^{-1}\,\rho_\beta(g_{\alpha\beta}g_{\alpha\beta}^{-1})\rho_\gamma g_{\alpha\beta\gamma}$$

$$= ag_\alpha^{-1}\rho_\beta\rho_\gamma\, g_{\alpha\beta\gamma} = ag_\alpha^{-1}\,\rho_{\beta\gamma}\, g_{\alpha(\beta\gamma)} = a\phi_{\alpha,\beta\gamma}.$$

Clearly, any two elements a,b in the same S_α are $\theta(1,0)$-related. To see that the $\theta(1,0)$-classes of S are precisely the family $\{S_\alpha : \alpha \in \Gamma\}$, take any $a \in S_\alpha$ and $b \in S_\beta$, for some $\alpha,\beta \in \Gamma$ such that $(a,b) \in \theta(1,0)$. Then for all $\gamma \in \Gamma$ and for all $x \in S_\gamma$ we have that

$$x\,\phi_{\gamma,\alpha} = x\,\phi_{\gamma,\beta} .$$

This implies in particular that $\gamma\alpha = \gamma\beta$, and that

$$g_\gamma^{-1}\,\rho_\alpha g_{\gamma\alpha} = g_\gamma^{-1}\,\rho_\beta g_{\gamma\beta} .$$

If we now post-multiply both sides of the above equation by the inverse of $g_{\gamma\alpha} = g_{\gamma\beta}$ and pre-multiply both sides by g_γ, then we have $\rho_\alpha = \rho_\beta$. Since the representation ρ is an embedding, we must have $\alpha = \beta$ and so $S_\alpha = S_\beta$. It follows then that $S/\theta(1,0)$ is isomorphic to Γ.

We point out also that by choosing a different set of bijections in place of $\{g_\alpha : \alpha \in \Gamma\}$, one can produce a different associative binary operation on the set S. In fact if S is finite, then we can determine the exact number of distinct associative binary operations that can be defined on the same set S in this way. If we repeat this process on the semigroup so constructed, we can construct other new semigroups. It follows, by induction on i and j, that for every ordered pair (i,j) there exists a semigroup S such that $S/\theta(i,j)$ is isomorphic to Γ. ❑

Example 1:2.9 Embed the two element left zero band $\Gamma = \{\alpha,\beta\}$ into the semigroup $\mathcal{T}(\Gamma^{(1)})$ of all full transformations of $\Gamma^{(1)}$ under the right regular representation. Then Γ is isomorphic to the semigroup $\{\rho_\alpha, \rho_\beta\}$, where

$$\rho_\alpha : \Gamma^{(1)} \to \Gamma^{(1)};\ \alpha \mapsto \alpha,\ \beta \mapsto \beta,\ 1 \mapsto \alpha$$

and

$$\rho_\beta : \Gamma^{(1)} \to \Gamma^{(1)};\ \alpha \mapsto \alpha,\ \beta \mapsto \beta,\ 1 \mapsto \beta.$$

Let the set $S_\alpha = \{a,b,c\}$ and $S_\beta = \{d,e,f\}$, and define the bijections g_α and g_β, respectively, as follows:

$g_\alpha : \Gamma^{(1)} \to S_\alpha;\ \alpha \mapsto a,\ \beta \mapsto b,\ 1 \mapsto c$ and $g_\beta : \Gamma^{(1)} \to S_\alpha;\ \alpha \mapsto d,\ \beta \mapsto e,\ 1 \mapsto f.$

The set $\Gamma \times \Gamma$ has four distinct elements and so there will be exactly four maps:

$\phi_{\alpha,\alpha} : S_\alpha \to S_{\alpha\alpha} = S_\alpha\ \ a \mapsto a,\ b \mapsto b,\ c \mapsto a;$ $\phi_{\alpha,\beta} : S_\alpha \to S_{\alpha\beta} = S_\alpha\ a \mapsto a,\ b \mapsto b,\ c \mapsto b;$

$\phi_{\beta,\beta} : S_\beta \to S_{\beta\beta} = S_\beta\ \ d \mapsto d,\ e \mapsto e,\ f \mapsto e;$ and $\phi_{\beta,\alpha} : S_\beta \to S_{\beta\alpha} = S_\beta\ d \mapsto d,\ e \mapsto e,\ f \mapsto d.$

Define a binary operation on $S = S_\alpha \cup S_\beta$ as described in Construction 1:2.7 and Theorem 1:2.6, and we have the following Cayley table:

	a	b	c	d	e	f
a	a	a	a	a	a	a
b	b	b	b	b	b	b
c	a	a	a	b	b	b
d	d	d	d	d	d	d
e	e	e	e	e	e	e
f	d	d	d	e	e	e

This semigroup is clearly not a band, and is not even an inflation of a left zero band. It does, however, satisfy the identity $zxy = zx$. ❑

Example 1:2.10 Adjoin an identity element to a semigroup S and denote the resulting semigroup by $S^{(1)}$. Then define a binary operation on the set

$$T = S^{(1)} \times S = \{(a,b):\ a \in S^{(1)},\ b \in S\} \quad \text{by} \quad (a,b)\otimes(c,d) = (ad,bd).$$

For any elements (a,b), (c,d) and (e,f) of T,

$$[(a,b)\otimes(c,d)]\otimes(e,f) = (ad,bd)\otimes(e,f) = ((ad)f,(bd)f) = (a(df),b(df))$$
$$= (a,b)\otimes(cf,df) = (a,b)\otimes[(c,d)\otimes(e,f)];$$

and hence $(T,\otimes)$ forms a semigroup. Now, take any $\theta^T(1,0)$-related elements, say (a,b) and (c,d). Then for all $(x,y) \in T$ we have

$$(x,y)\otimes(a,b) = (x,y)\otimes(c,d).$$

This implies that $(xb,yb) = (xd,yd)$, and, that in turn implies that $xb = xd$ for all x in $S^{(1)}$. But since $S^{(1)}$ is a monoid it follows that $b = d$, and hence the quotient $T/\theta^T(1,0)$ is isomorphic to S. ❑

The concepts of left and right *engamorphic products* were first introduced in the author's M.Sc. thesis (1991). Take a semigroup $(S,\bullet)$ and any endomorphism ϕ with the property that $(x\phi)\phi = x\phi$ for every x. It can be shown that the binary operation defined by $a \circ b = a\bullet(b\phi)$ is associative; and the semigroup $(S,\circ)$ (alternatively, written as $(S,\bullet,\phi;l)$) is called the *left engamorphic product of* $(S,\bullet)$ *with respect to* ϕ. The *right engamorphic product* $(S,\bullet,\phi;r)$ is defined by duality. Historically speaking, the concept of engamorphic products in Kopamu (1991) led the author to the idea of the family of congruences $\theta(n,m)$.

Remark 1:2.11 Consider the semigroup $(T,\otimes)$ constructed in Example 1:2.10. We claim that $(T,\otimes)$ is an engamorphic product on the direct product $(T,\bullet) = S^{(1)} \times S$ with respect to the endomorphism ϕ defined by $(x,y)\phi = (y,y)$. Take any elements (x,y) and (w,t) of $S^{(1)} \times S$. To show that ϕ is indeed a homomorphism,

$$((x,y)\bullet(w,t))\phi = (xw,yt)\phi = (yt,yt) \quad \text{and} \quad (x,y)\phi \bullet (w,t)\phi = (y,y)\bullet(t,t) = (yt,yt).$$

And since

$$(x,y)\bullet(w,t)\phi = (x,y)\bullet(t,t) = (xt,yt) = (x,y)\otimes(w,t),$$

we have that $(T,\otimes)$ is indeed the left engamorphic product on $S^{(1)} \times S$ with respect to ϕ. ❑

Henceforth, we will refer to the semigroup constructed in Example 1:2.10 as the *natural left engamorphic product* on $S^{(1)} \times S$, in contrast to the well known *direct product*. At times, we will simply call it the *natural enga-product on* $S^{(1)} \times S$.

Remark 1:2.12 The product in Example 1:2.10 can alternatively be considered as a user friendly case of the more general construction given in Construction 1:2.7. The enga-products introduced above can be adapted for other kinds of algebras. For example, if $(R,+,\bullet)$ is a ring then on the set $R \times R$ one can define the following associative binary operations:

$$(a,b) \blacklozenge (c,d) = (a+d, b+d), \text{ and } (a,b) \heartsuit (c,d) = (a \bullet d, b \bullet d),$$

$$(a,b) \oplus (c,d) = (a+c, b+d) \text{ and } (a,b) \otimes (c,d) = (a \bullet c, b \bullet d).$$

The products $\oplus$ and $\otimes$ are the well known direct products. It can be shown that together with the enga-product $\heartsuit$, $(R \times R, \oplus, \heartsuit)$ forms a ring and so does $(R \times R, \oplus, \otimes)$. The system $(R \times R, \oplus, \otimes, \heartsuit, \blacklozenge)$ forms a mysterious algebra, possessing some nice relationships interrelating these operations. Consider, in the particular, the ring $(\mathbf{Z},+,\bullet)$ of all integers under the usual addition and multiplication of numbers. If we define the products $\oplus$, $\otimes$, $\heartsuit$ and $\blacklozenge$ as above on the set $\mathbf{Z} \times \mathbf{Z}$, then we have a concrete example of such an interesting algebra. ❑

Problem 1:2.13 *The author strongly believes that the methods used in this thesis, in particular the family of congruences $\theta(n,m)$ and the enga-products defined above, could be generalised or otherwise adapted, to study other types of algebras such as rings, nearings and semirings.* ❑

The results presented in this and the next chapter will appear in the journal *Communications in Algebra* (see Kopamu (1995(a)).

We end this chapter with some interesting observations which may lead to further research in the future, but which have nothing to do with the rest of this thesis.

By a *representation* $\rho: S \to \mathcal{T}(X)$, we mean a homomorphism from a semigroup S into the semigroup $\mathcal{T}(X)$ of all full transformations on a non-empty set X.

Observation 1:2.14 *Let* $\rho: S \to \mathbf{T}(X)$ *be a representation of* S. *Then on the Cartesian product* $X \times S = \{(x,s): x \in X, s \in S\}$, *the binary operation* $\otimes$ *defined by* $(x,s)\otimes(y,t) = (x\rho_t, st)$ *is associative. Moreover, if* S *is left reductive, then the quotient* $(X\times S,\otimes)/\theta(1,0)$ *is isomorphic to* S. *Example 1:2.10 above concerns a particular case of this when* $X = S^{(1)}$, and ρ *is taken to be the right regular representation of* S. ❑

Observation 1:2.15 *Let* S *be a semigroup and denote by* $\mathbf{T}^{(n,m)}(S)$ *the set of all maps of the type* $f: S^n \times S^m \to S^{n+1+m}$. *One can define the following map (not a representation):*

$$\rho : S \to \mathbf{T}^{(n,m)}(S) \ , s \mapsto \rho_s.$$

where

$$\rho_s : S^n \times S^m \to S^{n+1+m}, \qquad (x,y) \mapsto xsy.$$

It can be verified that $\ker\rho = \theta(n,m)$. ❑

Observation 1:2.16 *Let I be an ideal of a semigroup S. Consider the following family of semigroup congruences on S:*

$$I\theta(n,m) = \{(a,b)\in S\times S: uav = ubv \text{ for all } u \text{ in } I^n \text{ and } v \text{ in } I^m\}.$$

This thesis is concerned with the particular set of ideals $I = S^k, k \geq 1$. *It would be interesting to determine what could be done with these congruences.* ❑

CHAPTER 2

A FAMILY OF CLASS-INTERSECTION PRESERVING INJECTIVE MAPS

In this chapter we introduce a useful family of class-intersection preserving one-to-one maps on the lattice of all semigroup species. Recall that a species is a class of semigroups closed under the operation of taking homomorphic images. The set $\mathbf{S_p}$ of all semigroup species forms a complete lattice under class containment $\subseteq$. First, for any set $\{\mathcal{V}_i : i \in I\}$ of species, the intersection $\bigcap\{\mathcal{V}_i : i \in I\}$ is again a species. We also see that $\bigcap\{\mathcal{V} \in \mathbf{S_p} : \mathcal{V}_i \subseteq \mathcal{V} \text{ for all } i \in I\}$ is the least semigroup species containing the union of the members of $\{\mathcal{V}_i : i \in I\}$. For any class $\mathcal{V}$ of semigroups, and any $(n,m) \in \mathbf{N}^{[0]} \times \mathbf{N}^{[0]}$ we define the class of semigroups $\mathcal{V}^{(n,m)} = \{S : S/\theta(n,m) \in \mathcal{V}\}$. For the case n=1 and m=0 we have by Construction 1:2.7 that the class $\mathcal{V}^{(1,0)}$ is non empty; and dually, so is the class $\mathcal{V}^{(0,1)}$. By repeating this process on the classes so obtained, one can deduce that $\mathcal{V}^{(n,m)}$ is non-empty for every (n,m).

2:1 SPECIES OF SEMIGROUPS

Lemma 2:1.1 *For any class $\mathcal{V}$ of semigroups and any $(n,m) \in \mathbf{N}^{[0]} \times \mathbf{N}^{[0]}$ we have the following statements concerning $\mathcal{V}^{(n,m)}$:*

(a) If $\mathcal{V}$ is closed under homomorphic images, then so is $\mathcal{V}^{(n,m)}$.

(b) If $\mathcal{V}$ is closed under arbitrary direct products then so is $\mathcal{V}^{(n,m)}$.

(c) If $\mathcal{V}$ is closed under both homomorphic images and subsemigroups, then $\mathcal{V}^{(n,m)}$ is also closed under subsemigroups.

Proof. (a) For any $S \in \mathcal{V}^{(n,m)}$ and any homomorphism Ψ from S onto T, we have (from Theorem 1:1.4) that $T/\theta^T(n,m)$ is a homomorphic image of $S/\theta^S(n,m)$ and so it follows that the class $\mathcal{V}^{(n,m)}$ is closed under taking homomorphic images if $\mathcal{V}$ is.

(b) Let S be the direct product of an arbitrary family $\{S_\alpha: \alpha \in \Gamma\}$ of semigroups from $\mathcal{V}^{(n,m)}$. We have (from Lemma 1:1.9) that $S/\theta^S(n,m)$ is isomorphic to the direct product of the family $\{S_\alpha/\theta^{S_\alpha}(n,m) : \alpha \in \Gamma\}$; and since $\mathcal{V}$ is assumed to be closed under arbitrary direct products, $S/\theta^S(n,m) \in \mathcal{V}$. Thus S belongs to $\mathcal{V}^{(n,m)}$, proving that $\mathcal{V}^{(n,m)}$ is also closed under arbitrary direct products.

(c) Take any $S \in \mathcal{V}^{(n,m)}$, and denote the family of all $\theta(n,m)$-classes by $\{S_\alpha: \alpha \in \Gamma\}$, where $\Gamma = S/\theta^S(n,m) \in \mathcal{V}$. Then for any subsemigroup T of S the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\theta^S(n,m)^\natural"] & \Gamma = S/\theta^S(n,m) \\
\cup & \cup \\
T \arrow[r, "\varphi"] \arrow[dr, "\theta^T(n,m)^\natural"'] & \Lambda \arrow[d, "\Phi"] \\
 & T/\theta^T(n,m)
\end{tikzcd}$$

where

(i) $\Lambda = \{\alpha: T \cap S_\alpha \neq \varnothing,\ \alpha \in \Gamma\}$ is a subsemigroup of Γ.

(ii) $\varphi: T \to \Lambda$, $a \mapsto \alpha$, where $a \in T \cap S_\alpha \neq \varnothing$

(iii) $\Phi: \Lambda \to T/\theta^T(n,m)$, $\alpha \mapsto a\theta^T(n,m)$, where $a \in T \cap S_\alpha \neq \varnothing$.

We will show that both of the above two maps are homomorphisms.
Consider first the map

$$\varphi: T \longrightarrow \Lambda,\ a \mapsto \alpha, \text{ where } a \in T \cap S_\alpha \neq \varnothing.$$

For any $a,b \in T$, $a \in T \cap S_\alpha \neq \varnothing$ and $b \in T \cap S_\beta \neq \varnothing$. Since T is a subsemigroup and $\theta^S(n,m)$ is a congruence on S, we have $ab \in T \cap S_{\alpha\beta} \neq \varnothing$; and so φ is a homomorphism since:

$$(a\varphi)(b\varphi) = \alpha\beta = (ab)\varphi.$$

Next we will show that the map

$$\Phi: \Lambda \longrightarrow T/\theta^T(n,m),\ \alpha \mapsto a\theta^T(n,m), \quad \text{where } a \in T \cap S_\alpha \neq \varnothing,$$

is a homomorphism. We observe that the map Φ is well defined since $\theta^S(n,m) \cap (T \times T) \subseteq \theta^T(n,m)$. Now, take any $\alpha,\beta \in \Lambda$. By the definition of Λ, there exist elements $a,b \in T$ such that $a \in T \cap S_\alpha \neq \varnothing$ and $b \in T \cap S_\beta \neq \varnothing$.

Again, since T is a subsemigroup and $\theta^S(n,m)$ is a congruence on S, we have that $ab \in T \cap S_{\alpha\beta} \neq \varnothing$; and Φ is homomorphism since:

$$(\alpha)\Phi\,(\beta)\Phi = a\theta^T(n,m)\ \ b\theta^T(n,m)\ = (ab)\theta^T(n,m) = (\alpha\beta)\Phi.$$

Thus we have shown that $T/\theta^T(n,m)$ is a homomorphic image of Λ.

If $\mathcal{V}$ is closed under subsemigroups, then Λ is a member of $\mathcal{V}$. If $\mathcal{V}$ is also closed under homomorphic images, and since $T/\theta^T(n,m)$ is a homomorphic image of Λ, it follows that T is also contained in $\mathcal{V}^{(n,m)}$. ❑

Theorem 2:1.2 *We have the following statements about certain species of semigroups:*

(a) If $\mathcal{V}$ is a variety of semigroups then so is $\mathcal{V}^{(n,m)}$.

(b) If $\mathcal{V}$ is an existence variety of regular semigroups, then so is the class $\mathcal{RS} \cap \mathcal{V}^{(n,m)}$.

(c) If $\mathcal{V}$ is a generalised variety, then so is $\mathcal{V}^{(n,m)}$.

(d) If $\mathcal{V}$ is a pseudovariety, then so is $\mathcal{F}in \cap \mathcal{V}^{(n,m)}$.

Proof. (a) From Lemma 2:1.1 (a), (b) and (c), if $\mathcal{V}$ is closed under taking: homomorphic images, arbitrary direct products and subsemigroups, then so is $\mathcal{V}^{(n,m)}$.

(b) If $\mathcal{V}$ is an existence variety, then the class $\mathcal{RS} \cap \mathcal{V}^{(n,m)}$ is closed under homomorphic images and arbitrary direct products. To show that $\mathcal{RS} \cap \mathcal{V}^{(n,m)}$ is an e-variety we need only show that it is closed under regular subsemigroups.

In the proof of Lemma 2:1.1(c), assume that T is a regular subsemigroup of $S \in \mathcal{RS} \cap \mathcal{V}^{(n,m)}$. We see that Λ is regular since it is a homomorphic image of T, and, since Λ is a subsemigroup of $S/\theta(n,m) \in \mathcal{V}$, Λ must also be contained in the e-variety $\mathcal{V}$. We conclude that T is a member of $\mathcal{RS} \cap \mathcal{V}^{(n,m)}$ since $T/\theta(n,m)$ is a homomorphic image of Λ.

(c) Suppose that $\mathcal{V}$ is a generalised variety. Then from Result 1.1(iii) of Ash (1985) (see Theorem 0:3.1), there exists a directed family $\{\mathcal{A}_i : i \in F\}$ of varieties such that $\mathcal{V} = \bigcup\{\mathcal{A}_i : i \in F\}$. Then it follows that $\mathcal{V}^{(n,m)} = \bigcup\{\mathcal{A}_i^{(n,m)} : i \in F\}$, where each $\mathcal{A}_i^{(n,m)}$ is a variety by Part (a) of this result. To show that $\mathcal{V}^{(n,m)}$ is a generalised variety, we must show that $\{\mathcal{A}_i^{(n,m)} : i \in F\}$ is a directed family. For any i and j, there exists some $\mathcal{W} \in \{\mathcal{A}_i : i \in F\}$ such that both $\mathcal{A}_i$ and $\mathcal{A}_j$ are contained in $\mathcal{W}$. It is easy to see that both $\mathcal{A}_i^{(n,m)}$ and $\mathcal{A}_j^{(n,m)}$ are contained in $\mathcal{W}^{(n,m)}$. Hence $\mathcal{V}^{(n,m)}$ is also a union of some directed family of varieties.

(d) If $\mathcal{V}$ is a pseudovariety, then from Theorem 0:3.2, there exists a least generalised variety $\mathcal{W}$ whose finite members form $\mathcal{V}$. From (c) above, the class $\mathcal{W}^{(n,m)}$ is also a generalised variety. We know from Theorem 0:3.2, that pseudovarieties are characterised as the finite members of some generalised variety. We observe that the finite members of $\mathcal{W}^{(n,m)}$ is precisely the finite members of $\mathcal{V}^{(n,m)}$. That is, $\mathcal{Fin} \cap \mathcal{V}^{(n,m)} = \mathcal{Fin} \cap \mathcal{W}^{(n,m)}$ and so the class $\mathcal{Fin} \cap \mathcal{V}^{(n,m)}$ is a pseudovariety. ❑

2:2 LATTICES OF SEMIGROUP SPECIES

For any class $\mathcal{W}$ of semigroups, consider the family of semigroup classes given by:

(2:2.1) $$K(\mathcal{W}) = \{\mathcal{W}^{(n,m)} : (n,m) \in \mathbf{N}^{[0]} \times \mathbf{N}^{[0]}\}.$$

If the class $\mathcal{W}$ is a species, then so is every member of this family (from Lemma 2:1.1(a)); and in general, $\mathcal{W}^{(n,m)} \subseteq \mathcal{W}^{(s,t)}$ if $(n,m) \leq (s,t)$. If $\mathcal{W}$ is a structurally closed species, for example the variety of all semigroups, then $K(\mathcal{W})$ is a singleton set containing only $\mathcal{W}$. In certain other cases, for example when $\mathcal{W}$ consists entirely of reductive semigroups such as groups or inverse semigroups, $K(\mathcal{W})$ has infinite cardinality. The following ascending chain of members of $K(\mathcal{W})$ exists:

$$\mathcal{W} \subseteq \mathcal{W}^{(1,0)} \subseteq \mathcal{W}^{(2,0)} \subseteq \ldots \subseteq \mathcal{W}^{(n,0)} \subseteq \mathcal{W}^{(n,1)} \subseteq \cdots \subseteq \mathcal{W}^{(n,m)}.$$

By the *structural closure* of a class $\mathcal{V}$ we mean

$$\mathcal{V}^{(\infty,\infty)} = \{ S : S \in \mathcal{V}^{(n,m)} \text{ for some } (n,m) \in \mathbf{N}^{[0]} \times \mathbf{N}^{[0]} \};$$

and (from Theorem 2:1.2(a)) if $\mathcal{V}$ is a variety, then so is every member of $K(\mathcal{V})$. Note that $\mathcal{V}^{(\infty,\infty)}$ is in fact the union of all semigroups in the family of varieties that form $K(\mathcal{V})$. For any two members of $K(\mathcal{V})$, one can always find a third member which contains both, and so by Theorem 0:3.1 it forms a generalised variety. Any semigroup in $\mathcal{V}^{(\infty,\infty)}$ will be referred to as a *structurally-$\mathcal{V}$* semigroup. The set $K(\mathcal{V})$ does not in general form a sublattice of the lattice of all varieties of semigroups, for $K(\mathcal{V})$ may fail to be closed under taking varietal meets. For example, consider the trivial variety $\mathcal{T}$. Although the variety $\mathcal{N}_2$ of null semigroups is not a member of $K(\mathcal{T})$, that is, $\mathcal{N}_2$ cannot be expressed in the form $\mathcal{N}_2 = \mathcal{T}^{(i,j)}$ for some i and j, it can be shown however that $\quad \mathcal{N}_2 = \mathcal{T}^{(0,1)} \cap \mathcal{T}^{(1,0)}$.

The following method of obtaining new identities from given ones will be useful. For any $(n,m) \in \mathbf{N}^{[0]} \times \mathbf{N}^{[0]}$ and any family of semigroup identities,

(2:2.2) $$\{P_j(x_1,x_2,\ldots,x_{k_j}) = Q_j(x_1,x_2,\ldots,x_{k_j}) : j \in F\},$$

sandwich each identity occurring there by an n-letter word $\mathbf{u(n)}$ and an m-letter word $\mathbf{w(m)}$ as shown below:

(2:2.3) $$\mathbf{u(n)}\, P_j(x_1,x_2,\ldots,x_{k_j})\, \mathbf{w(m)} = \mathbf{u(n)}\, Q_j(x_1,x_2,\ldots,x_{k_j})\, \mathbf{w(m)},$$

such that the set of letters occurring in the words $\mathbf{u(n)}$ and $\mathbf{w(m)}$ are disjoint from each other, and disjoint also from the set $\{x_1,x_2, \ldots ,x_{k_j}\}$. Furthermore, we make the restriction that each letter that occurs in the words $\mathbf{u(n)}$ or $\mathbf{w(m)}$ must appear only once and that the words $\mathbf{u(0)}$ and $\mathbf{w(0)}$ denote the empty word. For reasons that will become clear later, we shall write the identity (2:2.3), alternatively, as

(2:2.4) $$(P_j(x_1,x_2, \ldots ,x_{k_j}) , Q_j(x_1,x_2, \ldots ,x_{k_j}))\in \theta(n,m);$$

and denote the family of identities so obtained from (2:2.2) by

(2:2.5) $$\{(P_j(x_1,x_2, \ldots ,x_{k_j}) , Q_j(x_1,x_2, \ldots ,x_{k_j}))\in \theta(n,m) : j \in F\}.$$

For example, consider the set of identities $[xy = yx, x = x^2]$. In the above notation,

$$\{(xy , yx)\in \theta(1,1), (x ,x^2)\in \theta(1,1)\} \text{ and } \{(xy , yx)\in \theta(1,0), (x , x^2)\in \theta(1,0)\}$$

would denote the sets of identities, respectively,

$$\{zxyk = zyxk, zxk = zx^2k\} \qquad \text{and} \qquad \{zxy = zyx, zx = zx^2\}.$$

Theorem 2:2.6 *For every $(n,m)\in \mathbf{N}^{[0]} \times \mathbf{N}^{[0]}$ and any variety $\mathcal{V}$ of semigroups, if*

(2:2.7) $$\mathcal{V}= [\{P_j(x_1,x_2, \ldots ,x_{k_j}) = Q_j(x_1,x_2, \ldots ,x_{k_j}) : j \in F\}]$$

then

(2:2.8) $$\mathcal{V}^{(n,m)} = [\{(P_j(x_1,x_2, \ldots ,x_{k_j}) , Q_j(x_1,x_2, \ldots ,x_{k_j}))\in \theta(n,m) : j \in F\}].$$

Moreover, the map $\mathcal{V} \mapsto \mathcal{V}^{(n,m)}$ is both one to one and class-intersection preserving. It embeds $([\mathcal{T},\mathcal{V}],\cap)$ into the interval $([\mathcal{T}^{(n,m)},\mathcal{V}^{(n,m)}],\cap)$, where $\mathcal{T}$ denotes the trivial variety. In particular, if $\mathcal{V}$ is a structurally closed variety, then $\mathcal{V}= \mathcal{V}^{(n,m)}$.

Proof. It can be shown (see Kopamu (1991)) that a semigroup S satisfies the identity

$$(P_j(x_1,x_2, \ldots ,x_{k_j}) , Q_j(x_1,x_2, \ldots ,x_{k_j}))\in \theta(1,0)$$

if and only if $S/\theta(1,0)$ satisfies the identity

$$P_j(x_1,x_2, \ldots ,x_{k_j}) = Q_j(x_1,x_2, \ldots ,x_{k_j}).$$

Hence, it follows that the equality below holds:

$$\mathcal{V}^{(1,0)} = [(P_j(x_1,x_2, \ldots ,x_{k_j}) , Q_j(x_1,x_2, \ldots ,x_{k_j}))\in \theta(1,0)) : j \in F].$$

This, in fact, is the case when $n = 1$ and $m = 0$ of (2:2.8). We will prove the general case by inductive arguments. Now, fix $m = 0$ and for some $n = k$ suppose that (2:2.8) holds. That is, we are supposing that

$$\mathcal{V}^{(k,0)} = [(P_j(x_1,x_2,\ldots,x_{k_j}), Q_j(x_1,x_2,\ldots,x_{k_j}))\in\theta(k,0)) : j\in F],$$

where $\mathcal{V}$ is as given in (2:2.7). It is easy to verify that a semigroup S satisfies the set of identities

$$\{(P_j(x_1,x_2,\ldots,x_{k_j}), Q_j(x_1,x_2,\ldots,x_{k_j}))\in\theta(k+1,0)) : j\in F\}$$

if and only if $S/\theta(1,0)$ satisfies the set of identities

$$\{(P_j(x_1,x_2,\ldots,x_{k_j}), Q_j(x_1,x_2,\ldots,x_{k_j}))\in\theta(k,0)) : j\in F\},$$

which by assumption determine the variety $\mathcal{V}^{(k,0)}$. Hence we have the equality:

$$\mathcal{V}^{(k+1,0)} = [(P_j(x_1,x_2,\ldots,x_{k_j}), Q_j(x_1,x_2,\ldots,x_{k_j}))\in\theta(k+1,0)) : j\in F].$$

Thus it follows that for every $n \geq 0$, we have

$$\mathcal{V}^{(n,0)} = [(P_j(x_1,x_2,\ldots,x_{k_j}), Q_j(x_1,x_2,\ldots,x_{k_j}))\in\theta(n,0)) : j\in F].$$

Now, fix n and by inductive arguments on m this time, one can show in a dual way that for every $m \geq 0$:

$$\mathcal{V}^{(n,m)} = [(P_j(x_1,x_2,\ldots,x_{k_j}), Q_j(x_1,x_2,\ldots,x_{k_j}))\in\theta(n,m)) : j\in F].$$

Now, the map $\mathcal{V}\mapsto\mathcal{V}^{(n,m)}$ preserves set intersections, since for any varieties of semigroups $\mathcal{V}$ and $\mathcal{W}$ we have:

$$\begin{aligned}(\mathcal{V}\cap\mathcal{W})^{(n,m)} &= \{S: S/\theta(n,m)\in\mathcal{V}\cap\mathcal{W}\}\\ &= \{S: S/\theta(n,m)\in\mathcal{V} \text{ and } S/\theta(n,m)\in\mathcal{W}\}\\ &= \{S: S/\theta(n,m)\in\mathcal{V}\}\cap\{S: S/\theta(n,m)\in\mathcal{W}\}\\ &= \mathcal{V}^{(n,m)}\cap\mathcal{W}^{(n,m)}.\end{aligned}$$

Next we prove that $\mathcal{V}\mapsto\mathcal{V}^{(n,m)}$ is one to one. Consider the case n=1 and m=0. Suppose, for a contradiction, that there exist varieties $\mathcal{V}\neq\mathcal{W}$, such that $\mathcal{V}^{(1,0)} = \mathcal{W}^{(1,0)}$. Then without loss of generality, there exists a semigroup in $\Gamma\in\mathcal{V}$ but $\Gamma\notin\mathcal{W}$. By the method of Construction 1:2.7, there exists at least one semigroup S such that $\Gamma = S/\theta(1,0)$. By the assumption that $S/\theta(1,0) = \Gamma\notin\mathcal{W}$, the semigroup $S\notin\mathcal{W}^{(1,0)}$; but this same semigroup is contained in the variety $\mathcal{V}^{(1,0)}$ since $S/\theta(1,0) = \Gamma\in\mathcal{V}$. By this contradiction, we must have $\mathcal{V} = \mathcal{W}$; and it follows then that since $\mathcal{V}\mapsto\mathcal{V}^{(1,0)}$ is both one to one, and also preserves set intersections, it is therefore an $\cap$-embedding on the lattice of all varieties of semigroups. By applying the method of Construction 1:2.7 and its dual method repeatedly on G one can obtain a semigroup T such that $T/\theta(n,m)$ is isomorphic to S, and it follows that the map $\mathcal{V}\mapsto\mathcal{V}^{(n,m)}$ is one-to-one in general.

2:3 LATTICES OF EXISTENCE VARIETIES

Existence varieties of regular semigroups and bivarieties of orthodox semigroups also form species of semigroups, but because of the presence of an additional feature — a unary operation — a slightly different approach is required to prove results analogous to those proved in the last section. We will therefore consider them separately.

For a regular semigroup S we have (from Kopamu (1991)) that the left semigroup digraph $G=(\nabla(S),M(S))$ (defined in Section 1:2) has the following additional properties:

(2:3.1) For any $\alpha,\beta,\gamma \in \Gamma$, if $\alpha\beta = \alpha\gamma$ then $\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$. Equivalently, if $arr(S_\alpha,S_\beta)$ is non-empty, then $|arr(S_\alpha,S_\beta)| = 1$.

(2:3.2) For all $\alpha\in \Gamma$ we have $arr(S_\alpha,S_\alpha) \neq \varnothing$ and $\phi_{\alpha,\alpha'\alpha}$ is the identity map on S_α , where $\alpha' \in V(\alpha)$.

Because of the above properties, identities satisfied by semigroups in $\mathcal{RS}\cap\mathcal{V}^{(n,m)}$ can be obtained from those satisfied by semigroups in $\mathcal{V}$in the following way.

Take a (unary) semigroup identity, say

(2:3.3) $P(x_1,x_2,x_3,\ldots,x_k) = Q(x_1,x_2,x_3,\ldots,x_k)$.

If the identity is formed by words beginning with the same letter, then keep the identity unchanged. Otherwise, if (2:3.3) is non trivial, then form a new identity by adding a new letter on the left of both the words as follows:

(2:3.4) $x_0P(x_1,x_2,x_3,\ldots,x_k) = x_0Q(x_1,x_2,x_3,\ldots,x_k)$.

If, however, the identity (2:3.3) is the trivial identity, namely x=y, then form the identity xy =x (left zero).

Denote the new identity so obtained by

(2:3.5) $(P(x_1,x_2,x_3,\ldots,x_k) , Q(x_1,x_2,x_3,\ldots,x_k))\in \theta^*(1,0)$,

and denote the identity obtained by the dual method by:

(2:3.6) $(P(x_1,x_2,x_3,\ldots,x_k) , Q(x_1,x_2,x_3,\ldots,x_k)) \in \theta^*(0,1)$.

Since the method we have just described preserves identities formed by words that begin and end with the same letter, the only interesting cases will be the following:

We observe that $\mathcal{T}$ is the least subvariety of every variety $\mathcal{V}$, and so every subvariety of $\mathcal{V}$ occurs in the interval $[\mathcal{T},\mathcal{V}]$. Thus the image of this interval under the map $\mathcal{X} \mapsto \mathcal{X}^{(n,m)}$ lies in the interval $[\mathcal{T}^{(n,m)},\mathcal{V}^{(n,m)}]$. ❑

It is shown in Lemma 4:3.5 that the map $\mathcal{V} \mapsto \mathcal{V}^{(n,m)}$ does not map lattice intervals *onto* lattice intervals. However, we still do not know whether or not it preserves varietal joins. It is shown in Corollary 4:4.14 that when these maps are restricted to the subvarietiies of a semigroup variety which consists entirely of groups, they are onto, and thereby forcing the maps to preserve varietal joins as well.

If $\mathcal{V}$ is a species then so is $\mathcal{V}^{(n,m)}$ from Lemma 2:1.1(a). The same proof above works for the next two corollaries, in showing that the map $\mathcal{V} \mapsto \mathcal{V}^{(n,m)}$ is both one-to-one and preserves class-intersections.

Corollary 2:2.9 *The map* $\mathcal{V} \mapsto \mathcal{V}^{(n,m)}$ *is a* $\bigcap$*-embedding on the lattice of all semigroup species.* ❑

Corollary 2:2.10 *(i) The map* $\mathcal{V} \mapsto \mathcal{V}^{(n,m)}$ *is a* $\bigcap$*-embedding on the lattice of all generalised varieties.*

(ii) The map $\mathcal{V} \mapsto \mathcal{F}in \bigcap \mathcal{V}^{(n,m)}$ *is a* $\bigcap$*-embedding on the lattice of all pseudovarieties.* ❑

Problem 2:2.11 The problem of determining whether or not the map $\mathcal{V} \mapsto \mathcal{V}^{(n,m)}$ preserves joins of varieties, pseudovarieties, generalised varieties or existence varieties of regular semigroups remains an unsolved mystery. What we do know about this map is: that it is one-to-one, that it preserves class intersections, and that it does not map lattice intervals onto lattice intervals in general. That is, the intervals $[\mathcal{A},\mathcal{B}]$ and $[\mathcal{A}^{(n,m)},\mathcal{B}^{(n,m)}]$ may not be isomorphic, as demonstrated in Lemma 4:3.5 for a particular case. But in many of the cases we have seen, the map seems to preserve varietal joins as well. ❑.

$$(P(x_1,x_2,x_3,\ldots,x_k), Q(x_1,x_2,x_3,\ldots,x_k)) \in \theta^*(n,m)$$

for $(n,m) \in \{0,1\} \times \{0,1\}$. Note that the case for (0,1) is the dual of what we have described above, while the case (1,1) can be obtained by applying the dual method, namely (1,0), on those identities obtained by (0,1).

Lemma 2:3.7 *Let* S *be a regular semigroup and let* $\Gamma = S/\theta(n,m)$, *where* $(n,m) \in \{0,1\} \times \{0,1\}$. *For each* $a \in S$, if $a\theta(n,m) = \alpha \in \Gamma$, *then*

$$V(a) = \{b \in S: \ b\theta(n,m) \in V(\alpha)\}.$$

Hence, $\theta(n,m) \subseteq \{(a,b) : V(a) = V(b)\}$.

Proof. Take any $b \in S$ such that $b\theta(n,m) \in V(\alpha)$. Then for any $a^* \in V(a)$ and $b^* \in V(b)$,

$$aba = (aa^*)^n(aba)\,(a^*a)^m = (aa^*)^n\, a(a^*a)^m = a$$

and

$$bab = (bb^*)^n\,(bab)\,(b^*b)^m = (bb^*)^n\, b\,(b^*b)^m = b;$$

and so $b \in V(a)$. Conversely, for every $b \in V(a)$, it is easily seen that $b\theta(n,m)$ is an inverse of $a\theta(n,m)$ in Γ. For any $(a,b) \in \theta(n,m)$, since $a\theta(n,m) = b\theta(n,m)$, we have $V(a) = V(b)$. ❑

Corollary 2:3.8 *For any regular semigroup S and any* $(n,m) \in \{0,1\} \times \{0,1\}$, *$\theta(n,m)$ is a unary congruence on S. That is, if* $(x,y) \in \theta(n,m)$, *then* $(x^*,y^*) \in \theta(n,m)$, *for some inverse unary operation * on S.*

Proof. If S is regular, then so is $\Gamma = S/\theta(n,m)$. Choose any inverse unary operation $'$ on Γ. Then for each $\alpha \in \Gamma$, there exists a unique $\alpha' \in V(\alpha)$ determined by this unary operation. For each $\alpha \in \Gamma$, fix any element s^* of S such that $s^*\theta(n,m) = \alpha'$. Then by Lemma 2:3.7, $s^* \in V(s)$, where $s\theta(n,m) = \alpha$. The assignment $s \mapsto s^*$ is a well defined inverse unary operation on S. Now, for any $(a,b) \in \theta(n,m)$, since $a\theta(n,m) = \gamma = b\theta(n,m)$, and if $\gamma \mapsto \gamma'$, then fix any element x in the $\theta(n,m)$-class denoted by γ'. Then since $V(a) = V(b)$ from Lemma 2:3.7, and by the definition of the inverse unary operation *, we have $a^* = x = b^*$, and hence $(a^*,b^*) \in \theta(n,m)$. ❑

Theorem 2:3.9 *For every existence variety of regular semigroups*

(2:3.10) $\qquad \mathcal{V} = [\{P_j(x_1,x_2,\ldots,x_{k_j}) = Q_j(x_1,x_2,\ldots,x_{k_j}) : j \in F\}],$

we have that for every $(n,m) \in \{0,1\} \times \{0,1\}$

(2:3.11) $\mathcal{RS} \cap \mathcal{V}^{(n,m)} = [\{(P_j(x_1,x_2,\ldots,x_{k_j}), Q_j(x_1,x_2,\ldots,x_{k_j})) \in \theta^*(n,m) : j \in F\}]$.

Moreover, the map $\mathcal{V} \mapsto \mathcal{RS} \cap \mathcal{V}^{(n,m)}$ *is a* $\cap$*-homomorphism on the lattice of all existence varieties.*
In particular, when $\mathcal{V} \mapsto \mathcal{RS} \cap \mathcal{V}^{(1,0)}$ *is restricted to the subvarieties of an existence variety consisting entirely of left reductive semigroups, it is a* $\cap$*-embedding.*

Proof. We will prove this result for the case $(n,m) = (1,0)$. Let $\mathcal{V}$ be an existence variety defined by (2:3.10), and let $S \in \mathcal{RS} \cap \mathcal{V}^{(1,0)}$. Consider a typical identity defining $\mathcal{V}$, say

(2:3.12) $$P(x_1,x_2,\ldots,x_k) = Q(x_1,x_2,\ldots,x_k).$$

For notational simplicity, we shall write this identity simply as $P = Q$. In the case where the words forming (2:3.12) begin with different letters, and other than the trivial identity $x = y$, the proof that S satisfies $(P,Q) \in \theta^*(1,0)$ is immediate. Suppose that the two sides of the identity $P = Q$ begin with the same letter x_0. Then $P = x_0P'$ and $Q = x_0Q'$, where

$$P' = P'(x_1,x_2,\ldots,x_k) \text{ and } Q' = Q'(x_1,x_2,\ldots,x_k).$$

Then for all x in S, and for all choices of elements x_i in S, for $i = 1,\ldots, k$, we have $xx_0P' = xx_0Q'$. We may choose $x = x_0x_0'$, where x_0' is an inverse of x_0, and deduce that

$$x_0P'(x_1,x_2,\ldots,x_k) = x_0x_0'x_0P'(x_1,x_2,\ldots,x_k) = x_0x_0'x_0Q'(x_1,x_2,\ldots,x_k)$$
$$= x_0Q'(x_1,x_2,\ldots,x_k).$$

Suppose next that (2:3.12) is the trivial identity $x = y$. Then S satisfies $zx = zy$. In particular, putting $y = z'z$ (which is possible since S is regular), we have that for all z,x in S, $zx = z(z'z) = z$. We have thus proved the containment:

(2:3.13) $$\mathcal{RS} \cap \mathcal{V}^{(1,0)} \subseteq [\{(P_j(x_1,x_2,\ldots,x_{k_j}), Q_j(x_1,x_2,\ldots,x_{k_j})) \in \theta^*(1,0) : j \in F\}].$$

Conversely, since $\mathcal{V}$ is an e-variety, the unary semigroup identities $xx'x = x$ and $x'xx' = x'$ must be in the family of identities that determine $\mathcal{V}$. Since both these identities have right-hand and left-hand sides beginning with the same letter, the method of obtaining new identities described in the paragraph preceding Lemma 2:3.7 leaves these identities unchanged, and so every semigroup in the equational class at the right hand side of (2:3.13) satisfies $xx'x = x$ and $x'xx' = x'$. Hence S is regular. It remains to show that $S/\theta(1,0) \in \mathcal{V}$. We will consider the three possible cases:

Suppose that S satisfies the identity $(P,Q) \in \theta^*(1,0)$ for every identity $P = Q$ such that $P = x_0P'$ and $Q = x_0Q'$. Then the identity $(P,Q) \in \theta^*(1,0)$ is equivalent to $P = Q$, and since S is thus assumed to satisfy $P = Q$ it follows immediately that $S/\theta(1,0)$ satisfies this identity.

If $P = Q$, is the trivial identity $x = y$, then $(P,Q) \in \theta^*(1,0)$ becomes $zx = z$. Thus $zx = zy$ for all z,x,y in S and $S/\theta(1,0)$ is trivial.

The final case is where P and Q begin with different letters, so that $(P,Q) \in \theta^*(1,0)$ becomes $xP = zQ$. Since S satisfies $zP = zQ$, it is clear by definition that $S/\theta(1,0)$ satisfies $P = Q$ and so belongs to $\mathcal{V}$. We have thus proved the equality (2:3.11) for the case (n,m) = (1,0). The cases (n,m) = (0,1) and (n,m) = (1,1) can be similarly proved.

Although the map $\mathcal{V} \mapsto \mathcal{RS} \cap \mathcal{V}^{(1,0)}$ preserves class-intersections, it is not one-to-one in general. This is because the trivial variety $\mathcal{T}$ and the variety of left zero bands are both mapped onto the e-variety of left zero bands under this mapping. However, if $\mathcal{V}$ is left reductive (for example any e-variety of inverse semigroups), then it can be shown that the map is also one- to-one. ❑

2:4 FREE SEMIGROUPS

For any variety $\mathcal{V}$ of semigroups and any non empty set X, by the *free $\mathcal{V}$-semigroup over* X we mean a pair $(F_X(\mathcal{V}),i)$, where $F_X(\mathcal{V}) \in \mathcal{V}$ and a mapping $i: X \rightarrow F_X(\mathcal{V})$, such that for any map $g : X \rightarrow S$, for any $S \in \mathcal{V}$, there exists a unique homomorphism $h: F_X(\mathcal{V}) \rightarrow S$ such that $i \circ h = g$.

Theorem 2:4.1 *For any variety $\mathcal{V}$ of reductive semigroups and any $(n,m) \in \mathbf{N}^{[0]} \times \mathbf{N}^{[0]}$, if $(F_X(\mathcal{V}^{(n,m)}),i)$ denotes the free $\mathcal{V}^{(n,m)}$-semigroup over a set X, then*

$$(F_X(\mathcal{V}^{(n,m)})/\theta^F(n,m),\ i \circ \theta^F(n,m))$$

denotes the free $\mathcal{V}$-semigroup over X.

Proof. For any map $g: X \rightarrow S$, and $S \in \mathcal{V}$, since $\mathcal{V} \subseteq \mathcal{V}^{(n,m)}$ there exists by the freeness of $F = F_X(\mathcal{V}^{(n,m)})$ in the variety $\mathcal{V}^{(n,m)}$ a unique map $h: F \rightarrow S$ such that $i \circ h = g$. Then from Theorem 1:1.4 there exists a homomorphism

$$k: F/\theta^F(n,m) \rightarrow T/\theta^T(n,m), \quad \text{where } T = (F)h \subseteq S,$$

such that $\theta^F(n,m) \circ k = h \circ \theta^T(n,m)$. But since T is reductive, we have $\theta^F(n,m) \circ k = h$. Moreover, we also have $i \circ \theta^F(n,m) \circ k = g$. From the way we defined the class $\mathcal{V}^{(n,m)}$, the quotient $F/\theta^F(n,m)$ is contained in the variety $\mathcal{V}$. It follows therefore that the pair $(F/\theta^F(n,m), i \circ \theta^F(n,m))$ is indeed the free $\mathcal{V}$-semigroup over X. ❑

Problem 2:4.2 *Determine whether the congruence $\theta(i,j)$ is fully invariant for each i and j, in the sense that for any $\theta(i,j)$-related elements a and b, the elements $a\phi$ and $b\phi$ are also $\theta(i,j)$-related for all endomorphism ϕ in S.* ❑

Problem 2:4.3 *Is there a reductive semigroup for which $S^2 \neq S$?*

This problem was answered in positive by the Examiner, who pointed out that the free monogenic semigroup S is reductive but $S^2 \neq S$.

CHAPTER 3

THE THEORY OF STRUCTURALLY REGULAR SEMIGROUPS

In this chapter we introduce the class of structurally regular semigroups. Examples of such semigroups are presented, and relationships with other known generalisations of the class of regular semigroups are explored. Some fundamental results and concepts about regular semigroups are generalised to this new class. In particular, a version of Lallement's Lemma is proved for structurally regular semigroups. For the class of structurally orthodox semigroups, the least inverse semigroup, the least group congruence, and the maximum idempotent $\theta(n,m)$-class separating congruence are determined.

Recall that an element x of a semigroup is regular if there exists an (inverse) element y such that $xyx = x$ and $yxy = y$; and semigroups consisting entirely of such elements are called *regular*. Regular semigroups have received wide attention (see for example, Petrich (1984), Howie (1976) and Higgins (1991)). In the literature, the set of all inverses of a regular element x is denoted by $V(x)$. An element x is said to be an *idempotent* if $x^2 = x$; and semigroups consisting entirely of idempotent elements are called *bands*. *Inverse semigroups* are just the regular semigroups with commuting idempotents, or equivalently, they are regular semigroups with unique inverses. Regular semigroups with a unique idempotent element are easily seen to be *groups;* semigroups that are unions of groups are called *completely regular;* and regular semigroups whose idempotent elements form a subsemigroup are called *orthodox.*

The very first class of semigroups to be studied was the class of groups, and some of the important results in semigroup theory came about as a result of attempting to generalise results from group theory. For example, the Vagner-Preston Representation Theorem for inverse semigroups was influenced by Cayley's Theorem for groups. In the quest to generalise group-theoretic results, inverse semigroups quickly emerged as the most natural class to study, and even today inverse semigroups continue to receive what is arguably more than their fair share of attention. From about 1970 onwards, T.E. Hall and others began the attempt to generalise results on inverse semigroups to orthodox and regular semigroups. They characterised the least inverse semigroup congruence on orthodox semigroups, and hence T.E. Hall was able to prove a structure theorem for orthodox that generalises Munn's theorem for inverse semigroups. The trend towards greater generality has in turn led to the study of various generalisations of regular semigroups, and, in keeping with this trend, we here introduce a new class of semigroups, much larger than the class of regular semigroups, and different from any of the known generalisations. In fact it is shown that the class of all structurally regular semigroups (defined below) is different from each of the following: eventually regular semigroups, locally regular semigroups, nilpotent extensions of regular semigroups, and weakly regular semigroups.

We have already encountered some interesting properties of the family of congruences: $\{\theta(n,m) : n \geq 0 \text{ and } m \geq 0\}$. In Chapter 2 we showed that for any semigroup *species* — a class of semigroups closed under homomorphic images say C — the class $C^{(n,m)}$ of all semigroups S such that $S/\theta(n,m)$ belongs to C, also forms a species. The class of all regular semigroups forms an important species. A semigroup S is said to be *structurally regular* if there exists some ordered pair of non-negative integers (n,m) such that $S/\theta(n,m)$ is regular. For any class C of regular semigroups, we say that a semigroup S is a *structurally (n,m)-C* semigroup if $S/\theta(n,m)$ belongs to C, and more generally, semigroups in the class $C^{(\infty,\infty)} = \{S: S/\theta(n,m) \in C, \text{ for some } (n,m)\}$ will be called *structurally-C semigroups*.

In this chapter we will lay the foundations for a unified approach to the study of structurally regular semigroups, as a natural generalisation from the concept of regularity. We will therefore establish notations and concepts, with the aim of placing this new class of semigroups within the framework of classical semigroup theory. In particular we will be concerned with the classes of semigroups consisting of the following types. A semigroup S is said to be *structurally [orthodox, band, completely regular, inverse]* if and only if $S/\theta(n,m)$ is [orthodox, band, completely regular, inverse] for some (n,m).

After providing many examples and methods of constructing structurally regular semigroups in Section 3:1, we present in Section 3:2 a generalisation of the Lallement Lemma. In Section 3:3 we raise some open questions, and summarise the relationships that exist between the different classes of semigroups that generalise the concept of regularity. Then in Section 3:4 we determine the maximum idempotent $\theta(n,m)$-class separating congruence, the least inverse semigroup congruence, and the least group congruence on certain classes of structurally orthodox semigroups. The chapter ends in Section 3:5 by demonstrating how regular semigroups could be studied using our family of congruences.

In the subsequent Chapters 4, 5, 6 and 7, we will describe the lattice structures of some semigroup varieties formed entirely by structurally regular semigroups. We point out that examples of structurally regular semigroups have appeared in the literature under different names. For example: Gerhard has in (1977a) and (1977b) studied the lattices of certain varieties of structurally band semigroups; Bogdanovic and Stamenkovic (1988) studied nilpotent extensions of semilattices of right groups; Higgins in (1984) determines identities of certain structurally regular semigroups; and inflations of completely regular semigroups were studied by Clarke (1981) where he provides a method of obtaining a set of identities that determine varieties formed by such semigroups. And Petrich (1974) determined completely the lattices formed by certain varieties consisting entirely of 2-nilpotent extensions of orthodox normal bands of groups.

The results presented in Sections in 3:1, 3:2 and 3:3 form a paper titled 'The concept of structural regularity' which has been submitted to the journal *Portugaliae Mathematica* (see Kopamu 1995(b)); while those in Section 3:5 were published in the *Bulletin of the Southeast Asian Mathematical Society* (see Kopamu (1994)); and the results in Section 3:4 have not been published.

3:1 EXAMPLES OF STRUCTURALLY REGULAR SEMIGROUPS

We first give a more useful characterisation of structurally regular semigroups.

Theorem 3:1.1 *Let (n,m) be an ordered pair of non-negative integers. For any semigroup* S, $S/\theta(n,m)$ *is regular (and hence,* S *is structurally regular) if and only if for each element a in* S *there exists a' such that*

$$zaa'aw = zaw \text{ and } za'aa'w = za'w \quad \text{for all } z \in S^n \text{ and } w \in S^m.$$

Proof. For each element a of a semigroup S, denote $a\theta(n,m)$ by α. Then $S/\theta(n,m)$ is regular if and only if for every a there exists b such that $b\theta(n,m) = \beta$, $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$, that is, if and only if for every a in S there exists b such that $(aba,a) \in \theta(n,m)$ and $(bab,b) \in \theta(n,m)$, that is, if and only if for every a in S there exist a' = b in S such that for all z in S^n and w in S^m, zaa'aw = zaw and za'aa'w = za'w. ❑

Example 3:1.2 Take any nontrivial k-nilpotent semigroup N, any regular semigroup R, and consider the direct product $S = N \times R$. Then for any element $s = (n,r) \in N \times R$, define $s' = (0,r')$, where 0 is the zero element of N, and r' is an inverse of r in the regular semigroup R. Then for all $z = (0,y) \in S^k = \{(0,y) : y \in R^k\}$, we have

$$zs = (0,y)(n,r) = (0, yr) = (0n0n, yrr'r) = (0,y)(n,r)(0,r')(n,r) = zss's$$

and

$$zs' = (0,y)(0,r') = (0, yr') = (00n0,yr'rr') = (0,y)(0,r')(n,r)(0,r') = zs'ss';$$

which proves that $S/\theta(0,k)$ is regular. Hence, by Theorem 3:1.1, $S = N \times R$ is structurally regular. ❑

The condition that for each element a there exists b such that zaw = zabaw for all z in S^n and w in S^m implies that there exists an element, namely $a^* = bab$, such that $zaw = zaa^*aw$ and $za^*w = za^*aa^*w$. Other examples of structurally regular semigroups are presented in Examples 1:1.7, 1:1.8 and 1:2.9. In fact, the method of construction described in Example 1:2.10 can be used to construct more such examples.

P.M. Edwards (1983) defined a semigroup S to be *eventually regular* if for each x in S there exists some positive integer n such that x^n is regular. In Munn (1961) the inverses of the regular element x^n are referred to as the *pseudoinverses* of x in the case where x^n is contained in a group.

Example 3:1.3 Take any nontrivial k-nilpotent semigroup N and consider the semigroup $S = N^{(1)}$, the semigroup obtained from N by adjoining an identity element. Clearly, for each element x in S, the k-th power x^k is either the zero element of the nilpotent semigroup or the adjoined identity element. Thus, S is eventually regular. However since S is a monoid, it is reductive and so $S/\theta(i,j) = S$ for every ordered pair (i,j). Hence, eventual regularity does not imply structural regularity. ❑

It is shown in Example 3:3.1 that the class of all structurally regular semigroups is not contained in the class of all eventually regular semigroups. However, for the cases considered in Lemma 3:1.4 below, every structurally regular semigroup is necessarily eventually regular.

Denote by Reg(S) the set of all regular elements of S, that is

$$\mathrm{Reg}(S) = \{x \in S: xx'x = x \text{ for some } x' \in S\};$$

and by $E_{(n,m)}(S)$ we mean the union of all its idempotent $\theta(n,m)$-classes of S:

$$E_{(n,m)}(S) = \{x: (x,xx) \in \theta(n,m)\}.$$

We shall say an element x is (n,m)-*idempotent* if it is $\theta(n,m)$ related to some idempotent element, that is, if $ux^2v = uxv$ for all u and v in S^n and S^m, respectively. We will demonstrate in this chapter that the concept of (n,m)-idempotent elements is analogous the concept of idempotents elements in regular semigroups. In fact, as shown in Theorem 3:1.10, if $S/\theta(n,m)$ is an orthodox semigroup, then $E_{(n,m)}$ forms a subsemigroup of S. We will simply denote by E(S) the set of all idempotent elements of S (rather than $E_{(0,0)}(S)$).

Lemma 3:1.4 *Let* S *be a semigroup. If* $S/\theta(n,m)$ *satisfies* $x = x^{k+1}$ *for some positive integer* k, *then* S *satisfies* $x^{(n+1+m)} = x^{(n+1+m)(k+1)}$, $k \geq 1$. *Hence, if* $\mathcal{V}$ *is a variety consisting entirely of completely regular semigroups, then* $\mathcal{V}^{(n,m)}$ *consists entirely of eventually regular semigroups.*

Proof. Suppose that $S/\theta(n,m)$ satisfies an identity of the form $x^{k+1} = x$, for some $k \geq 1$. Then for each element a of S, $(a,a^{k+1}) \in \theta(n,m)$. This implies that for all $u \in S^n$ and $v \in S^m$, $uav = ua^{k+1}v$.

In particular, $a^{n+1+m} = a^{n+k+1+m}$. Now, putting $b = a^{n+1+m}$, we see that

$$\begin{aligned} b^{k+1} = (a^{n+1+m})^{k+1} &= a^{(n+1+m)(k+1)} \\ &= a^{(n+1+m)}a^{k(n+1+m)} \\ &= a^{(n+k+1+m)}a^{k(n+m)} \end{aligned}$$

$$= a^{(n+1+m)}a^{k(n+m)}$$

$$= a^{(n+k+1+m)}a^{k(n+m-1)}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$= a^{n+k+1+m}$$

$$= b;$$

Consider the element $b = a^{n+1+m}$. If $k>1$ then we have $b(b^{k-1})b = b$, and if $k = 1$, then $bbb = b$ since in this case $bb = b$. In any case b is regular, and hence it follows that S is eventually regular. Now, if $\mathcal{V}$ is a variety consisting entirely of completely regular semigroups, then as shown in Corollary 14 of Higgins (1984), every semigroup in $\mathcal{V}$ satisfies an identity of the form $x^{k+1} = x$, for some $k \geq 1$. Then from what we have just proved, the class $\mathcal{V}^{(n,m)}$ consists of eventually regular semigroups. ❑

An element x is said to be a *weak inverse* (see Page 537 of Pin and Thérien (1993)) of y if $xyx = x$. This does not, in general, imply that $yxy = y$ but of course x is a regular element. We dub the semigroups consisting entirely of such elements as a *weakly regular* semigroups. That is, a semigroup is weakly regular if for each element a of S there exists b such that $bab = b$. We point out that the semigroup in Example 3:1.3 above is one such example. For, if we put $x' = 1$ when x is the identity element, and put $x' = 0$ otherwise, then it is easy to verify that $x'xx' = x'$. This then establishes the fact that the class of all structurally regular semigroups does not even contain the class of all weakly regular semigroups. In fact any semigroup with a zero element is weakly regular.

Lemma 3:1.5 *If* $S/\theta(n,m)$ *is regular [orthodox] then the set* Reg(S) *of all regular elements of* S *forms a regular [orthodox] subsemigroup of* S.

Proof. Suppose that $\Gamma = S/\theta(n,m)$ is regular. For any elements a,b of Reg(S), let $a\theta(n,m) = \alpha$ and $b\theta(n,m) = \beta$. Then since Γ is regular, there exists $\gamma \in \Gamma$ such that $(\alpha\beta)\gamma(\alpha\beta) = \alpha\beta$. Now, take any $c \in S$ such that $c\theta(n,m) = \gamma$, and any inverses $a' \in V(a)$ and $b' \in V(b)$. Then since $(ab, abcab) \in \theta(n,m)$,

$$(ab)c(ab) = (aa')^n [(ab)c(ab)] (b'b)^m = (aa')^n [ab] (b'b)^m = ab;$$

and so $ab \in$ Reg(S), proving that Reg(S) forms a regular subsemigroup. Now, suppose that $S/\theta(n,m)$ is orthodox, and take any idempotents $e,f \in E(S)$. Then there exist $\alpha^2 = \alpha$ and $\beta^2 = \beta$ in Γ such that $e\theta(n,m) = \alpha$ and $f\theta(n,m) = \beta$. Since $E(\Gamma)$ forms a band, by assumption, $(\alpha\beta)^2 = \alpha\beta$. Therefore, $(ef,(ef)^2) \in \theta(n,m)$, and $(ef)^2 = e^n(ef)^2f^m = e^n(ef)f^m = ef$. Thus E(S) forms a subsemigroup, and hence Reg(S) is orthodox. ❑

For an ideal extension S of a semigroup K by T, if there exists a homomorphism ϕ from S onto K such that $a\phi = a$ for every a in K, and the Rees quotient S/K is isomorphic to T, then such an extension is called a *retract extension* (See Petrich (1973)). A retract extension by an n-nilpotent semigroup is called an *n-inflation*. The following result characterises n-inflations of regular semigroups.

Theorem 3:1.6 *The following statements are equivalent:*

(i) S *is a (n+1)-nilpotent extension of a regular subsemigroup, and there exists a regular-element-separating congruence* γ *on* S *with the property that every* γ*-class contains a regular element.*

(ii) S *is an (n+1)-inflation of Reg(S).*

(iii) For each element a *of* S *there exists* a^* *such that for all* $x \in S^n$,

$$xa = xaa^*a \quad \text{and} \quad aa^*ax = ax.$$

(iv) Both $S/\theta(n,0)$ *and* $S/\theta(0,n)$ *are regular semigroups.*

Proof. (i)$\Rightarrow$ (ii). Suppose that (i) holds, and define ϕ to be the map which sends each element x to the unique regular element contained in the γ–class that contains x. Then (ii) holds.

(ii)$\Rightarrow$ (iii). We are supposing that there exists a retract endomorphism $\phi : S \rightarrow R$, where R is a regular ideal of S and where S/R is a (n+1)-nilpotent semigroup. If $x \in S^n$ then $xa \in S^{n+1} \subseteq R$, and so $xa = (xa)\phi$. If $a^* \in V(a\phi)$ in R then $a^*\phi = a^*$, and so

$$xa = (xa)\phi = (x\phi)(a\phi) = (x\phi)(a\phi)\,(a^*\phi)\,(a\phi) = (xa)\phi\,(a^*a)\phi = xaa^*a.$$

Similarly, $ax = aa^*ax$ for all x in S^n. It is clear that Reg(S) = R, a subsemigroup.

(iii)$\Rightarrow$ (i) Suppose that (iii) holds in S and consider the congruence $\delta_n = \theta(n,0) \cap \theta(0,n)$. It is clear by the assumption that for each element a there exists a^* such that $(a,(aa^*)^na)$, $(a^*,(a^*a)^na^*) \in \delta_n$. In S the element $(aa^*)^na$ is regular since

$$[(aa^*)^na]\,a^*\,[(aa^*)^na] = (aa^*)^n\,(aa^*)^{n+1}a = (aa^*)^na,$$

by repeated use of the equality $xaa^*a = xa$. Thus every δ_n-class contains a regular element. Next, we will show that the congruence δ_n is regular-element-separating. Take any $(a,b) \in \delta_n \cap (\mathrm{Reg}(S) \times \mathrm{Reg}(S))$. Then for any inverse a' and b' of a and b, respectively, we have that

$$a = aa'a = (aa')^n a = (aa')^n b \quad \text{and} \quad b = bb'b = (bb')^n b = (bb')^n a$$

and so $(a,b) \in \mathcal{L}$ (Green's relation) in Reg(S). Then since a and b are regular, we have by Howie (1976) that there exists inverses a^* and b^* , respectively, of a and b such that $aa^* = bb^*$.

Therefore,

$$a = (aa^*)a = (bb^*)a = (bb^*)^n a = (bb^*)^n b = bb^*b = b.$$

We have thus shown that every δ_n-class contains a unique regular element. Hence the map $\phi : S \rightarrow S$, $a \mapsto (aa^*)^n a$ is well defined. For any inverse a' of a regular element a, we have $a = (aa')^n a = a\phi$, and so ϕ fixes regular elements. It follows that $\mathrm{Reg}(S) = S\phi$. We will show that ϕ is a homomorphism. Take any elements a and b in S, and denote simply by a° and b° the unique regular elements in the δ–classes, respectively, which contain a and b. Since the regular elements form a subsemigroup by Lemma 3:1.5, the element $a^\circ b^\circ$ is also a regular element. But since δ is a regular-element-separating congruence, $a^\circ b^\circ$ is the unique regular element in the δ-class which contains ab. That is, $a^\circ b^\circ = (ab)^\circ$. Therefore

$$(a\phi)(b\phi) = a^\circ b^\circ = (ab)^\circ = (ab)\phi$$

for all a,b in S, and so ϕ is indeed a homomorphism. We also see that $\mathrm{Reg}(S) \subseteq S^{n+1}$ since for each regular element s and any inverse s', we have $s = s(s's)^{n+1}$. The reverse inclusion $S^{n+1} \subseteq \mathrm{Reg}(S)$ also holds since for any elements $a_1, a_2, a_3, \ldots, a_n, a_{n+1}$ of S,

$$a_1a_2a_3 \ldots a_na_{n+1} = a_1a_2a_3 \ldots a_n(a_{n+1}\phi) \qquad (\text{since } (a_{n+1}, a_{n+1}\phi) \in \delta_n)$$

$$= (a_1\phi)(a_2\phi)(a_3\phi) \ldots (a_n\phi)(a_{n+1}\phi).$$

We have the last equality since $a_{n+1}\phi$ is contained in S^{n+1}, and since ϕ is a homomorphism it follows that $S^{n+1} \subseteq S^n$. Hence $S^{n+1} = \mathrm{Reg}(S)$. Thus we have shown that S is an (n+1)-nilpotent extension of the regular subsemigroup $S\phi = \mathrm{Reg}(S) = S^{n+1}$, which proves that (i) holds.

(iii) $\Leftrightarrow$ (iv). Clear from the definition of $\theta(n,0)$ and $\theta(0,n)$. ❑

Theorem 3:1.7 *Every n-inflation of a regular semigroup is structurally regular.* ❑

We point out that for the semigroup S given in Example 1:2.9, $S/\theta(1,0)$ is regular but $S/\theta(0,1)$ is not regular. Therefore the regularity of $S/\theta(n,0)$ does not, in general, imply the regularity of $S/\theta(0,n)$. Thus it follows, in view of

Theorem 3:1.6 (iv), that the class of all n-inflations of regular semigroups is properly contained in the class of all structurally regular semigroups.

Example 3:1.8 Consider the two element semilattice $A = \{a,0\}$. On the Cartesian product $S = A^{(1)} \times A = \{(x,y): x \in A^{(1)} \text{ and } y \in A\}$, where $A^{(1)}$ is the semigroup obtained by adjoining an identity element to A, define a binary operation $\otimes$ by $(a,b)\otimes(c,d) = (ad,bd)$. Then $(S,\otimes)$ forms a semigroup since it is the natural *enga-product* on $A^{(1)} \times A$; and $S/\theta(1,0)$ is isomorphic to the semilattice A (see Examples 1:2.10 and Remark 1:2.11). Therefore, S is structurally regular. However, for every positive integer n,

$$S^n = \{(a,a), (a,0), (0,0), (0,a)\}$$

is not regular, since the element (a,0) is not regular. Thus not every structurally regular semigroup is a nilpotent extension of a regular semigroup. ❑

In Example 3:1.17 it is shown that a nilpotent extension of a regular semigroup is not necessarily a structurally regular semigroup. Thus the concepts of structural regularity and the concept of nilpotent extension of regular semigroups are distinct.

Lemma 3:1.9 *If* $S/\theta(n,m)$ *is regular then every* $\theta(n,m)$*-class contains a regular element. Moreover, every element x of* S^{n+1+m} *can be expressed in the form* $x = abc$, *where* $a \in S^n$, $b \in \mathrm{Reg}(S)$ *and* $c \in S^m$.

Proof. Suppose that $S/\theta(n,m)$ is regular. Then for each element a in S there exists an element a' such that for all $u \in S^n$ and $v \in S^m$,

$$uav = uaa'av = u(aa')^n a(a'a)^m v;$$

and hence the elements a and $b = (aa')^n a(a'a)^m$ are $\theta(n,m)$-related. Since

$$ba'b = (aa')^n a(a'a)^m a'(aa')^n a(a'a)^m = (aa')^n a(a'a)^m = b,$$

b is a regular element. Now, take any element x in S^{n+1+m}. Then there exist elements $x_1,x_2,x_3, \ldots, x_{n+1+m}$ in S such that

$$x = (x_1x_2x_3 \ldots x_n)x_{n+1}(x_{n+2}x_{n+3}x_{n+4} \ldots x_{n+1+m}) = abc,$$

where $a = x_1x_2x_3 \ldots x_n$ $b = (x_{n+1}x_{n+1}')^n x_{n+1}(x_{n+1}'x_{n+1})^m$ and $c = x_{n+2}x_{n+3}x_{n+4} \ldots x_{n+1+m}$, and of course, b is a regular element that is $\theta(n,m)$-related to x_{n+1}. ❑

Theorem 3:1.10 *If* $S/\theta(n,m)$ *is orthodox, then* $E_{(n,m)}$ *forms a subsemigroup. In particular, if* $S/\theta(n,m)$ *is an inverse semigroup then the following equalities hold:*

$$\theta^{E_{(n,m)}}(n,m) = \theta^{S}(n,m) \cap (E_{(n,m)} \times E_{(n,m)}) \qquad (3:1.11)$$

$$\theta^{E_{(0,0)}}(n,m) = \theta^{S}(n,m) \cap (E_{(0,0)} \times E_{(0,0)}) \qquad (3:1.12)$$

$$\theta^{Reg(S)}(n,m) = \theta^{S}(n,m) \cap (Reg(S) \times Reg(S)) \qquad (3:1.13)$$

Proof. If $S/\theta(n,m)$ is regular, then by Lemma 3:1.5, Reg(S) forms a subsemigroup of S. Suppose that $S/\theta(n,m)$ is orthodox, and take any $x,y \in E_{(n,m)}$. Then $x\theta(n,m)$ any $y\theta(n,m)$ are idempotent elements of $S/\theta(n,m)$. Hence $(xy)\theta(n,m)$ is also an idempotent and so $xy \in E_{(n,m)}$, proving that $E_{(n,m)}$ forms a subsemigroup. To prove that (3:1.11) holds, take any $(a,b) \in \theta^{E_{(n,m)}}$, and let $u \in S^n$ and $v \in S^m$.
Then

$$uav = ua^naa^mv = ua^nba^mv = uabav = uab^2av = uba^2bv$$

$$= ubabv = ub^nab^mv = ub^nbb^mv = ub^{n+1+m}v = ubv,$$

and so $(a,b) \in \theta^{S}(n,m) \cap (E_{(n,m)} \times E_{(n,m)})$. Since the reverse containment holds trivially, the equality (3:1.11) follows. One can show, in particular, that (3:1.12) also holds,

To prove (3:1.13), take any $(a,b) \in \theta^{Reg(S)}(n,m)$, and let

$$d = (aa')^n, \quad e = (a'a)^m, \quad f = (bb')^n \quad \text{and} \quad g = (b'b)^m,$$

where $a' \in V(a)$, $b' \in V(b)$. We see that d, e, f, $g \in Reg(S)$ since they are all idempotent elements. Now, for all u in S^n, and v in S^m,

$$uav = udaev = udbev \qquad (\text{since } (a,b) \in \theta^{Reg(S)}(n,m))$$

$$= udfbgev = ufdbegv = ufdaegv = ufagv = ufbgv = ubv.$$

We have the fourth equality since $S/\theta(n,m)$ is inverse, so the idempotent $\theta(n,m)$-classes commute (see Theorem 0:2.5). Thus $(a,b) \in \theta^{S}(n,m)$. Since the reverse containment holds trivially, the equality (3:1.13) follows. ❑

Theorem 3:1.14 *Every engamorphic product of a regular semigroup is structurally regular.*

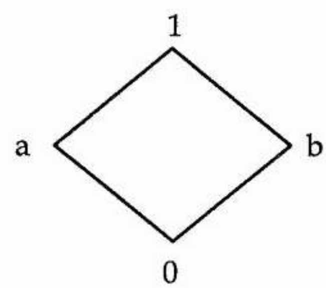
Proof. Suppose that $(S,\bullet)$ is regular, and take any retractive endomorphism ϕ. Then for each $a \in S$ fix any inverse of the regular element $a\phi$ when considered as an element in the regular semigroup $(S,\bullet)$, and denote it by $a^* = (a\phi)'$. Then for all s in S, we have by the retractive nature of ϕ that

$$s\circ a = s\bullet(a\phi) = s\bullet(a\phi)\bullet(a\phi)'\bullet(a\phi) = s\circ a\circ a^{*}\circ a;$$

and $(S,\circ)/\theta(1,0)$ is regular. Hence $(S,\circ) = (S,\bullet,\phi;1)$ is structurally regular. ❑

The example below shows that the concepts of *engamorphic-products on regular semigroups,* and the concept of *inflations of regular semigroups* are distinct. In fact, in view of Theorem 3:1.14, the class of all engamorphic products on regular semigroups forms a proper subclass of the class of structurally regular semigroups.

Example 3:1.15 Consider the 4-element semilattice $(S,\bullet) = \{1,a,b,0\}$, forming the diamond shown below, and define a map ϕ from S into itself which sends $1\mapsto 1$, $a\mapsto 1$, $b\mapsto 0$, and $0\mapsto 0$.



Define a binary operation on the set S by $x\circ y = x\bullet(y\phi)$.

$\circ$	1	a	b	0
1	1	1	0	0
a	a	a	0	0
b	b	b	0	0
0	0	0	0	0

The semigroup $(S,\bullet,\phi;1) = (S,\circ)$ is not regular since the element b is not regular. Moreover, since $S^k = \{1,a,b,0\} = S$ for all $k \geq 1$, the semigroup $(S,\circ)$ is not a nilpotent extension of a regular semigroup. This proves that an engamorphic product on a regular semigroup may not necessarily be an ideal extension of some regular semigroup. Hence S can not be, in this particular case, an infaltion of a regular semigroup. ❑

Remark 3:1.16 In view of Remark 1:2.11, the semigroup constructed in Example 3:3.1 is an engamorphic product of a regular semigroup. Since that semigroup is not eventually regular, we conclude that taking of engamorphic products does not necessarily produce eventually regular semigroups.

3:2. A GENERALISATION OF LALLEMENT'S LEMMA

If $S/\theta(n,m)$ is regular then for each element x of S one can define the following set:

(3:2.1) $\quad V_S(x;n,m) = \{y\colon\ uxyxv = uxv \text{ and } uyxyv = uyv,\ u \in S^n \text{ and } v \in S^m\}$.

We will call each member of the above set an *(n,m)-inverse* of x. In particular, if the element x is regular, then the set of all its inverses coincides with the set $V_S(x;0,0)$ (see Lemma 2:3.7). And of course we have $V_S(x;0,0) \subseteq V_S(x;n,m)$.

For any semigroup S, any ordered pair (n,m), and for all u in S^n and v in S^m we have the following additional concepts: an element x is called *(n,m)-idempotent* if $ux^2v = uxv$, and the set of all such elements is denoted by $E_{(n,m)}(S)$. Semigroups which consist entirely of such (n,m)-idempotent elements will be called *(n,m)-bands*. The concept of *(0,0)-band, (0,0)-idempotent,* and *(0,0)-inverse* coincide with the usual meaning of the words *band, idempotent*, and *inverse*, respectively.

A semigroup will be called *(n,m)-orthodox* if $S/\theta(n,m)$ is orthodox. Equivalently, these are structurally regular semigroups for which the union of all idempotent $\theta(n,m)$-classes forms a subsemigroup. In this section, we demonstrate that (n,m)-idempotents behave in a way somewhat similar to the way in which idempotent elements do. In fact for any element x' of $V_S(x;n,m)$ one can show that both xx' and x'x are (n,m)-idempotents.

We refer the reader to (0:1.1) - (0:1.5) for the definitions of the five Green's relations $\mathcal{L},\mathcal{R},\mathcal{H},\mathcal{D},\mathcal{J}$. These relations have played an important role in our understanding of semigroup structures. The following new relations, which are in fact generalisations of the Green's relations, are quite useful in the study of structurally regular semigroups.

For any Green's relation $\mathcal{X} \in \{\mathcal{R},\mathcal{L},\mathcal{H},\mathcal{D},\mathcal{J}\}$, define a new relation $\mathcal{X}_{(n,m)}$ as follows: for any elements a,b of S, we say $(a,b) \in \mathcal{X}_{(n,m)}$ if and only if the classes $a\theta(n,m)$ and $b\theta(n,m)$ are $\mathcal{X}$-related in $S/\theta(n,m)$. For example, $(a,b) \in \mathcal{R}_{(n,m)}$ in S if and only if there exists $x,y \in S^{(1)}$ such that

$$b\theta(n,m) = a\theta(n,m)\, x\theta(n,m) \qquad \text{and} \qquad a\theta(n,m) = b\theta(n,m)\, y\theta(n,m).$$

This is equivalent to saying that the pairs (b,ax) and (a,by) are $\theta(n,m)$-related in S.

Theorem 3:2.2 *Take any structurally (n,m)-regular semigroup S, and any elements a and b. Then for any $a' \in V_S(a;n,m)$ and $b' \in V_S(b;n,m)$ the following statements hold:*

(i) $(a,b) \in \mathcal{L}_{(n,m)}$ in S if and only if there exist (n,m)-inverses a' and b' of a and b, respectively, such that (a'a,b'b) are $\theta(n,m)$-related.

(ii) $(a,b) \in \mathcal{R}_{(n,m)}$ in S if and only if there exists a' and b' such that (aa',bb') are $\theta(n,m)$-related.

(iii) $(a,b) \in \mathcal{H}_{(n,m)}$ in S if and only if there exists a' and b' such that (aa',bb') and (a'a,b'b) are $\theta(n,m)$-related pairs.

Proof. We prove only statement (i). The remaining statements can be proved similarly. Let $a\theta(n,m) = \alpha$, $b\theta(n,m) = \beta$, and suppose that $(a,b) \in \mathcal{L}_{(n,m)}$ in S. Then in the regular semigroup $S/\theta(n,m)$, $(\alpha,\beta) \in \mathcal{L}$. Hence by Lemma II:4.7 of Howie (1976) (see Theorem 0:2.1(i)) there exist $\alpha' \in V(\alpha)$ and $\beta' \in V(\beta)$ such that $\alpha'\alpha = \beta'\beta$. If a' and b' are in S such that $a'\theta(n,m) = \alpha'$, $b'\theta(n,m) = \beta'$, then $(a'a,b'b) \in \theta(n,m)$ as required. ❑

The following theorem is a useful generalisation of a result due to T.E. Hall (see for example, Exercise 14 on Page 55 of Howie (1976)), which, in turn, is a generalisation of Lallement's Lemma.

Theorem 3:2.3 *Let ϕ be a homomorphism from S onto T. If $S/\theta(n,m)$ is regular, then for any $t \in T$ and any $t' \in V_T(t;n,m)$ there exists $s' \in V_S(s;n,m)$ such that $(s\phi,t)$ and $(s'\phi,t')$ are $\theta(n,m)$-related pairs in T.*

Proof. We have by Theorem 1:1.4 that $T/\theta^T(n,m)$ is a homomorphic image of $S/\theta^S(n,m)$ under the map $\phi_{(n,m)}$: $a\theta^S(n,m) \mapsto a\phi\theta^T(n,m)$, for each element a of S. Hence, every $\theta^T(n,m)$-class is an image of some $\theta^S(n,m)$-class under $\phi_{(n,m)}$, and the quotient $T/\theta^T(n,m)$ is a homomorphic image of $S/\theta^S(n,m)$. Denote the $\theta(n,m)$-classes of S and T, respectively, as follows:

$$\{S_\alpha : \alpha \in \Gamma = S/\theta(n,m)\} \qquad \text{and} \qquad \{T_\alpha : \alpha \in \Lambda = T/\theta(n,m)\}.$$

Take any $t \in T_\alpha$, $\alpha \in \Lambda$, and any $t' \in T_{\alpha'}$, where α' is a inverse of the regular element α. Then by Hall's generalisation of Lallement's Lemma, and by the commutativity of the diagram in Theorem 1:1.4, there exist elements β and β' in Γ such that $(\beta)\phi_{(n,m)} = \alpha$ and $(\beta')\phi_{(n,m)} = \alpha'$. This means that here exists s and s' in the $\theta^S(n,m)$-classes S_β and $S_{\beta'}$ respectively, such that $s\phi \in T_\alpha$ and $s'\phi \in T_{\alpha'}$. ❑

It is known that Lallement's lemma does not hold true in arbitrary semigroups. In fact, this lemma fails to hold in the semigroup of all positive integers under addition, since it does not have any idempotent element but the entire semigroup can be mapped onto a trivial semigroup, which of course is an idempotent. The following result is in fact a generalisation of the Lallement Lemma up to structurally regular semigroups.

Corollary 3:2.4 *Let ϕ be a homomorphism from S onto T. If $S/\theta(n,m)$ is regular, then for each idempotent f of T, there exists an idempotent element e of S such that $e\phi = f$.*

Proof. Since ϕ is onto, there exists some $a \in S$ such that $a\phi = f$. Take any $x \in V_S(a^2;n,m)$ and consider $e = (axa)^{n+1+m}$. We will show that e is an idempotent of S such that $e\phi = a\phi = f$. It is not difficult to see that (axa) is $\theta(n,m)$-related to $(axa)^i$ in S for every $i \geq 1$.

Now,

$$\begin{aligned} e^2 &= (axa)^{n+1+m}\,(axa)^{n+1+m} \\ &= (axa)^n\,[(axa)^{1+m}\,(axa)^{n+1}]\,(axa)^m \\ &= (axa)^n\,[\,axa\,]\,(axa)^m \quad (\text{since } (axa\,,\,(axa)^{1+m}\,(axa)^{n+1}) \in \theta(n,m)) \\ &= (axa)^{n+1+m} = e; \end{aligned}$$

and

$$\begin{aligned} e\phi &= ((axa)^{n+1+m})\phi = ((axa)\,(axa)^{n+m-1}\,(axa))\phi \\ &= (a\phi)\,[xa\,(axa)^{n+m-1}\,ax]\phi\,(a\phi) \\ &= (a\phi)^{n+2}\,[xa\,(axa)^{n+m-1}\,ax]\phi\,(a\phi)^{m+2} \\ &= (a^{n+2}\,[xa\,(axa)^{n+m-1}\,ax]\,a^{m+2})\phi \\ &= (a^{n+2}\,[x]\,a^{m+2})\phi \qquad (\text{since } x \in V_S(a^2;n,m)) \\ &= (a^{n+2}\,[x]\,a^{m+2})\phi = (a^n\,a^2xa^2\,a^m)\phi \\ &= (a^{n+2+m})\phi = (a\phi)^{n+1+m} = a\phi = f. \quad \square \end{aligned}$$

The following result shows that if T is a homomorphic image of some structurally regular semigroup S, then every regular element of T is a homomorphic image of some regular element of S.

Corollary 3:2.5 *Let ϕ be a homomorphism from S onto T. If $S/\theta(n,m)$ is regular, then*

$$\mathrm{Reg}(T) = (\mathrm{Reg}(S))\phi.$$

Proof. Clearly $\mathrm{Reg}(T) \supseteq (\mathrm{Reg}(S))\phi$. To prove the reverse containment, take any $x \in \mathrm{Reg}(T)$. Now for any $x' \in V(x)$, the elements xx' and $x'x$ are both idempotents in T. By Corollary 3.24 there exists idempotent elements e and f in S such that $e\phi = xx'$ and $f\phi = x'x$. Since ϕ is onto, there exists a in S such that $a\phi = x$. Since $S/\theta(n,m)$ is regular, we have by Lemma 3:1.9 that there exists (at least one) regular element d in Reg(S) such that a and d are $\theta(n,m)$-related. Therefore, since Reg(S) forms a subsemigroup,

$$c = eaf = e^n a f^m = e^n d f^m = edf \in \mathrm{Reg}(S).$$

We also observe that

$$c\phi = (ebf)\phi = (e\phi)(b\phi)(f\phi) = (xx')(x)(x'x) = x.$$

Thus we have shown that $\mathrm{Reg}(T) \subseteq (\mathrm{Reg}(S))\phi$ and so the equality holds. ❑

3:3 RELATIONSHIPS WITH OTHER GENERALISATIONS OF REGULAR SEMIGROUPS

The following counter example proves that the class of all structurally regular semigroups is not contained in the class of all eventually regular semigroups. Combining that with Example 3:1.3, we conclude that these two classes are not comparable; that is, neither contains the other.

Example 3:3.1 Let N denote the set of all positive integers, and consider the semigroup $S = N \times N$ with the multiplication $\blacksquare$ given by

(3:3.2) $$(n,m) \blacksquare (p,q) = (n-m+\max(m,p)\,,\, q-p+\max(m,p)).$$

This is the so-called *bicyclic* semigroup, which plays an important role in the theory of inverse semigroups. Now, consider $T = S^{(1)} \times S$, where S^1 denotes the semigroup obtained by adjoining an identity element **1** to S, and define a multiplication $\blacklozenge$ on T as follows:

(3:3.3) $$x \blacklozenge y = [a,b] \blacklozenge [c,d] = [a \blacksquare d,\, b \blacksquare d]\,,\ x = [a,b]\,,\ y = [c,d] \in T = S^{(1)} \times S.$$

More precisely,

$$x \blacklozenge y = \begin{cases} [(r-s+\max(s,k)\,,\, l-k+\max(k,s))\,,\,(t-u+\max(u,k)\,,\, l-k+\max(k,u)]. \\ \qquad \text{if } x = [(r,s),(t,u)] \text{ and } y = [(i,j),(k,l)] \text{ or } y = [\mathbf{1},(k,l)], \\ [(k,l)\,,\,(t-u+\max(u,k)\,,\, l-k+\max(k,u)]. \\ \qquad \text{if } x = [\mathbf{1},(t,u)] \text{ and } y = [(i,j),(k,l)] \text{ or } y = [\mathbf{1},(k,l)] \end{cases}$$

From Example 1:2.10, $(T,\blacklozenge)$ is the natural left engamorphic product on $T = S^{(1)} \times S$, and so is a semigroup. Moreover, $T/\theta(1,0)$ is isomorphic to S. Hence, $(T,\blacklozenge)$ is structurally regular. We will prove that it is not eventually regular. First we note that

(3:3.4) $$\mathrm{Reg}(T) = \{\, [(a,b),(c,d)] \in T:\ b \geq d\,,\, b,d \in N\}$$

To see this, take any regular element, say $x = [(a,b),(c,d)]$ of $(T,\blacklozenge)$. Then by assumption there exists an element, say $x' = [(e,f)\,,\,(g,h)]$ such that $x \blacklozenge x' \blacklozenge x = x$ and $x' \blacklozenge x \blacklozenge x' = x'$. This implies that the following equalities hold in the bicyclic semigroup:

(3:3.5) $$(c,d) \blacksquare (g,h) \blacksquare (c,d) = (c,d) \qquad \text{and } (g,h) \blacksquare (c,d) \blacksquare (g,h) = (g,h)$$

(3:3.6) $$(a,b) \blacksquare (g,h) \blacksquare (c,d) = (a,b)$$

From (3:3.5), we have by the uniqueness of inverses in S that (g,h) = (d,c); and by substituting that equality into (3:3.6) we have

(3:3.7) $$(a,b)\blacksquare(d,c)\blacksquare(c,d) = (a,b)$$

But since $(d,c)\blacksquare(c,d) = (d,d)$, it follows that $a - b + \max(b,d) = a$, and we have that $b \geq d$.

Conversely, it is straightforward, but tedious, to verify that for any $y = [(n,m),(p,q)]$ with $m \geq q$, the element $y' = [(r,s), (q,p)]$, $s \geq p$, is an inverse of y. Thus the set in (3:3.4) gives all the regular elements of $(T,\bullet)$.

To show that T is not eventually regular, consider the element $x = [\mathbf{1}, (1,2)]$, where **1** is the adjoined identity element of S^1.
Then

$$x^2 = [(1,2), (1,3)],\ x^3 = [(1,3), (1,4)],\ x^4 = [(1,4), (1,5)] \ldots.$$

In general,

$$x^k = [(1,k), (1,k+1)] \text{ for } k \geq 2.$$

Since k is never greater than k+1, x^k does not belong to Reg(T). Hence, x is not eventually regular. Thus T is not eventually regular, but of course it is structurally regular. ❑

A semigroup S is said to be *locally regular* if for every idempotent e of S, the subsemigroup $eSe = \{exe: x \in S\}$ is regular. The following result gives the relationship between *locally regular semigroups* and *structurally regular* semigroups.

Lemma 3:3.8 *Every structurally regular semigroup is locally regular.*

Proof. Suppose that $S/\theta(n,m)$ is regular, and take any $x \in eSe$, $e \in E(S)$. Then for any $x' \in V(x;n,m)$, $x = exe = e^n x e^m = e^n(xx'x)e^m = xx'x$. By straightforward verification, one can show that the element $x^* = e(x'xx')e \in eSe$ is indeed an inverse of x, and so x is therefore regular. Hence eSe is a regular subsemigroup, proving that S is locally regular. ❑

The next example shows that the converse of Lemma 3:3.8 does not hold, and hence the containment is strict.

Example 3:3.9 Let N be the semigroup of all positive integers under addition, G be a non trivial group, and $\phi : N \rightarrow G$ be the constant map which sends every element of N to the identity element e of G. Denote by $(S,\bullet)$ the ideal extension of G by N with respect to the homomorphism ϕ. Then the multiplication $\bullet$ on S is as follows:

$$x \bullet y = \begin{cases} x\ , & \text{if} \quad x \in G,\quad y \in N \\ y\ , & \text{if} \quad x \in N,\quad y \in G \\ & \\ xy\ , & \text{otherwise} \end{cases}$$

The identity element e of G becomes the unique idempotent element of S. Since eSe = eGe = G, (S,$\bullet$) is a locally regular semigroup. However, we see that (S,$\bullet$) is not structurally regular since for every (i,j) and any element x of N, $x\theta(i,j)$ forms a singleton set, and that the element x is not regular in S. Thus not every locally regular semigroup is structurally regular. ❑

A semigroup S is said to be *locally eventually regular* if for every idempotent e of S, the subsemigroup eSe = {exe: $x \in S$} is eventually regular; and S is *structurally eventually regular* if $S/\theta(n,m)$ is eventually regular for some (n,m). It is clear that both the classes of all structurally regular semigroups and the class of all eventually regular semigroups are strictly contained in the class of all structurally eventually regular semigroups. And a semigroup is called *structurally locally eventually regular* if $S/\theta(n,m)$ is locally eventually regular for some (n,m).

Lemma 3:3.10 *Every structurally eventually regular semigroup is locally eventually regular.*

Proof. Suppose that S is structurally eventually regular, and consider any idempotent element $e \in E(S)$. Then $S/\theta(n,m)$ is eventually regular for some ordered pair of non negative integers (n,m). For each $x \in eSe$ there exists an element b of S, and a positive integer k such that (x^kbx^k,x^k) and (bx^kb,b) are $\theta(n,m)$-related pairs in S. This implies that $u\, x^kbx^k\, v = ux^kv$ and $ubx^kbv = ubv$ for all $u \in S^n$ and $v \in S^m$. We will show that $a = e(bx^kb)e \in eSe$ is an inverse of x^k. Now,

$$x^k = e\,(x^k)\,e = e^n\,(x^k)\,e^m = e^n\,(x^kbx^k)\,e^m$$

$$= e^n\,(x^kebex^k)\,e^m = e^n\,(x^ke\,(bx^kb)\,ex^k)\,e^m = x^k\,a\,x^k,$$

and

$$a = e(b)e = e^n\,(b)e^m = e^n\,(b\,x^kb)\,e^m = (e^n\,be)\,x^k\,(e\,b\,e^m) = ax^ka;$$

and hence it follows that S is locally eventually regular. ❑

The previously encountered Example 3:3.9 also serves to show that the converse of Lemma 3:3.10 does not hold.

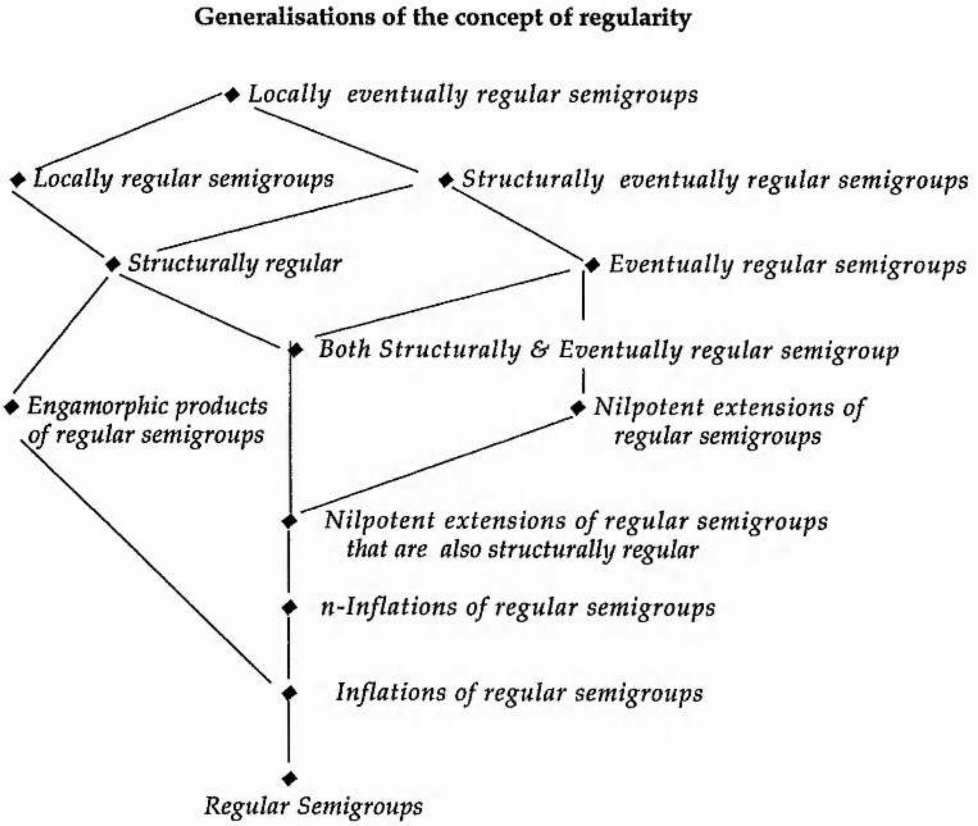


Figure 3:3.11 Strict containment relationships of some generalisations of regular semigroups.

Semigroup *species* are classes of semigroups closed under taking homomorphic images. In Figure 3:3.11, we summarise the containment relationships that exist between some known species of semigroups. In the diagram, a continuous line indicates a strict containment.

The semigroup (N,+) does not belong to any of the classes so far considered, although it appears as subsemigroup of some regular semigroups. Hence the classification presented in Figure 3:3.11 does not exhaust the class of all semigroups. However, the class of all finite semigroups is included here since they are eventually regular.

Lemma 3:3.12 *The class of all locally regular semigroups is structurally closed.*

Proof. Suppose that $S/\theta(n,m)$ is locally regular, and take any $e \in E(S)$. Then for any $x \in eSe$, $x\theta(n,m)$ is regular in $S/\theta(n,m)$ since $x\theta(n,m)$ is contained in the local subsemigroup of $S/\theta(n,m)$ with identity element $e\theta(n,m)$. Therefore, by assumption, there exists $a \in S$ such that (xax,x) and (axa,a) are $\theta(n,m)$-related pairs in S. Hence $uxaxv = uxv$ and $uaxav = uav$ for all u in S^n and v in S^m. Now, in S, $x = exe = e^nxe^m = e^nxaxe^m = xax$. It can be shown that $y = e(axa)e \in eSe$ is an inverse of x, and so S is locally regular. ❑

Lemma 3:3.13 *The class of all locally eventually regular semigroups is structurally closed.*

Proof. Suppose that $S/\theta(n,m)$ is locally eventually regular, and take any $e \in E(S)$. Then for any $x \in eSe$, there exists a positive integer $k \geq 1$, such that $x^k\theta(n,m)$ is regular in $S/\theta(n,m)$, since $x\theta(n,m)$ is contained in the local subsemigroup of $S/\theta(n,m)$ with identity element $e\theta(n,m)$. Therefore, by assumption, there exists $a \in S$ such that (x^kax^k,x^k) and (ax^ka,a) are $\theta(n,m)$-related pairs in S. Hence $ux^kax^kv = ux^kv$ and $uax^kav = uav$ for all u in S^n and v in S^m. Now, in S, $x^k = ex^ke = e^nx^ke^m = e^nx^kax^ke^m = x^kax^k$. It can be shown that $y = e(ax^ka)e \in eSe$ is an inverse of x^k, and so S is locally eventually regular. ❑

Lemma 3:3.14 *The class of all structurally regular semigroups and the class of all nilpotent extensions of regular semigroups are not comparable, in the sense that neither class contains the other.*

Proof. As shown in Theorem 3:1.8, a structurally regular semigroup may not be a nilpotent extension of a regular semigroup. Therefore, the class of all structurally regular semigroups is not contained in the class of all nilpotent extensions of regular semigroups. The converse also does not hold since by Example 5:1.1, a nilpotent extension of a regular semigroup may not be structurally regular. Therefore, these classes are not comparable in the sense that neither contains the other. ❑

Finally, we now demonstrate how one can produce concrete examples of semigroups from the types given in Figure 3:3.11. Let R_0 be a non trivial regular semigroup, say the bicyclic semigroup, and N be a non trivial nilpotent semigroup.

(i) Let $R_1 = R_0 \times N$ be the direct product of R_0 and N. Then, as shown in Example 3:1.2 R_1 is both a nilpotent extension and a structurally regular semigroup.

(ii) Let $R_2 = R_1^{(1)}$, be the semigroup obtained by adjoining an identity element to R_1. And as shown in Example 3:1.3, R_2 is eventually regular, but is neither structurally regular nor a nilpotent extension.

(iii) Let $R_3 = (R_2^{(1)} \times R_2, \Theta)$, where $R_2^{(1)}$ is the semigroup obtained by adjoining an identity element to R_2, $R_2^{(1)} \times R_2$ is the Cartesian product, and the multiplication Θ is defined as follows: $(a,b)\Theta(c,d) = (ad,bd)$. Then as was the case for the semigroup in Example 3:3.1, R_3 is a structurally eventually regular semigroup but is not eventually regular.

(iv) Let R_4 be the ideal extension of R_3 by the semigroup (N,+) of all positive integers under addition determined by a constant map which sends every element of N to a fixed idempotent element of R_3. Then every local subsemigroup of R_4 turns out to be a local subsemigroup of R_3. As was the case for the semigroup in Example 3:3.9, R_4 is not structurally eventually regular but is locally eventually regular.

(v) From Lemma 3:3.12 and Lemma 3:3.13 any structurally locally [eventually] regular semigroup is again locally [eventually] regular.

One can construct structurally regular semigroups using the method described in Example 1.2.10. Example 3:3.9 gives a locally regular semigroup that is not structurally regular. The construction of engamorphic products on a regular semigroup, or the taking of a nilpotent extension of a regular semigroup are well known procedures. Thus each of the classes given on Figure 3:3.11 are distinct and non empty.

We complete this section with a characterisation of permutative semigroups. A semigroup is said to be *permutative* if it satisfies a permutation identity. In particular a semigroup is said to be *commutative* if it satisfies the permutation identity $xy = yx$.

Theorem 3:3.14 *A semigroup S is permutative if and only if it is a structurally commutative semigroup.*

Proof. Suppose that $S/\theta(n,m)$ is commutative. Then it follows from Theorem 2:2.6 that for all $x, y \in S$, $u \in S^n$ and $v \in S^m$ $uxyv = uyxv$. Clearly, this is a permutation identity. To prove the converse, we need to show that for every permutative semigroup S, $S/\theta(n,m)$ is commutative for some (n,m). But that follows from a theorem of Putcha and Yaqub given below. ❑

Theorem 3:3.15 (Yaqub and Putcha (1971)) *Let S be a semigroup such that, for all $x_1, x_2, \ldots, x_n$ in S,*

(3:3.16) $$x_1x_2 \ldots x_n = x_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(n)} \qquad (n \geq 2)$$

where σ is a fixed permutation of $\{1,2, \ldots, n\}$ distinct from the identity permutation. Then there exists an integer k such that, for all $u, v \in S^k$ and for all $x_1, x_2 \in S$ we have that

$$ux_1x_2v = ux_2x_1v.$$

In particular S^k is medial. ❑

It is well known that any commutative regular semigroup is an inverse semigroup, since in this case the idempotent elements commute. The following analogous result holds for structurally regular semigroups.

Corollary 3:3.17 *Let S be a structurally regular semigroup. If S is permutative, then it is a structurally inverse semigroup.* ❑

Problem 3:3.18 *It is known that for every commutative regular semigroups, the idempotent elements are central, and hence such semigroups are strong semilattice of groups. Is there an analogous statement involving structurally regular semigroups which are permutative?* ❑

Problem 3:3.19 *Where does the class of all weakly regular semigrouops fit in the diagram of Figure 3:3.11?* ❑

The next result, though it has nothing to do with this present chapter, will be of use later in the thesis.

Lemma 3:3.20 *Let S be a regular semigroup. Then for any ordered pair (n,m) of non negative integers, the following statements hold:*

(i) $\theta(n,m) \subseteq \theta(1,1)$

(ii) $E_{(n,m)} = E(S)$

(iii) $V(a;n,m) = V(a)$ *for all* a *in* S.

Proof. (i) Clearly holds for regular semigroups since $S^k = S$ for all $k \geq 1$.

(ii) Take any element a in S and suppose that a and aa are $\theta(n,m)$-related. Then for any inverse a′ of a, we hae

$$a = aa'a = (aa')^n a(a'a)^m = (aa')^n(aa)(a'a)^m = (aa')(aa)(a'a) = aa$$

and so $E_{(n,m)} \subseteq E(S)$. The reverse containment holds trivially, and that proves (ii).

(iii) Take any (n,m)-inverse a* of an element a. Then since the element a is regular, there exists an inverse a′ of a. We will show that a* is also an inverse of a.

Now

$$a = aa'a = (aa')^n a(a'a)^m = (aa')^n(aa^*a)(a'a)^m = (aa')(aa^*a)(a'a) = aa^*a;$$

and similarly, $a^* = a^*aa^*$. This proves that $V(a;n,m) \subseteq V(a)$. The reverse containment holds trivially, and the equality holds. ❑

3:4 STRUCTURALLY ORTHODOX SEMIGROUPS

Structurally orthodox semigroups form an important class of structurally regular semigroups. Our purpose in this section will be to generalise some well known concepts and important results in the theory of orthodox semigroups up to structurally orthodox semigroups. In fact, some of the results and proofs presented in this section resemble very much some well known results about orthodox semigroups (see for example, Chapter VI of Howie (1976)). The following result is a generalisation of Theorem 0:2.4.

Theorem 3:4.1 *Let S be a structurally regular semigroup. Then the following statements are equivalent:*

(i) $S/\theta(n,m)$ *is orthodox*

(ii) For any elements a,b of S, $a' \in V_S(a;n,m)$, $b' \in V_S(b;n,m)$ we have that $b'a' \in V_S(ab;n,m)$.

(iii) For each $e \in E_{(n,m)}(S)$, *we have* $V_S(e;n,m) \subseteq E_{(n,m)}(S)$.

Proof. (i) $\Rightarrow$ (ii). Take any semigroup S and suppose that (i) holds. Then for any $a,b \in S$, any $a' \in V_S(a;n,m)$, $b' \in V_S(b;n,m)$, and for all $u \in S^n$, $v \in S^m$, we have:

$$u(ab)(b'a')(ab)v = u[(aa'ab)(b'a')(abb'b)v = ua(a'abb')^2bv$$

$$= u[a(a'abb')b]v = uabv,$$

and

$$u[(b'a')ab(b'a')]v = u[(b'bb'a')ab(b'a'aa')]v = u[b'(bb'a'a)^2a']v$$

$$= ub'(bb'a'a)a'v = ub'a'v;$$

and so $b'a'$ is an (n,m)-inverse of ab.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds, and take any $e \in E_{(n,m)}$ and any $x \in V_S(e;n,m)$. Let $u \in S^n$, $v \in S^m$.

Then

$$u(xex)v = uxv \qquad \text{and} \qquad uexev = uev.$$

The elements xe and ex are both (n,m)-idempotents, and so each is an (n,m)-inverse of itself. It follows by the assumption that $(ex)(xe) \in V_S((xe)(ex);n,m)$. i.e. $ex^2e \in V_S(x;n,m)$. Therefore,

$$uxv = ux(ex^2e)xv = u(xex)(xex)xv = ux^2v;$$

which proves that $V_S(e;n,m) \subseteq E_{(n,m)}(S)$ and so (iii) holds.

(iii) $\Rightarrow$ (i). Take any $e,f \in E_{(n,m)}(S)$, and let x be any (n,m)-inverse of ef. Then

$$u(ef)(fxe)(ef)v = u(efxef)v = u(ef)v$$

and $\quad u(fxe)ef(fxe)v = u(fxefxe)v = uf(xefx)ev = ufxev;$

and so ef $\in V_S(fxe;n,m)$. We need only show that fxe $\in E_{(n,m)}(S)$ to complete the proof since by assumption this will mean that ef is (n,m)-idempotent. Now,

$$u(fxe)^2v = u(fxefxe)v = u(fxe)v;$$

and thus fxe $\in E_{(n,m)}(S)$, proving that ef $\in E_{(n,m)}(S)$. It follows that the idempotent elements of $S/\theta(n,m)$ form a subsemigroup, and so $S/\theta(n,m)$ is orthodox. Equivalently, $E_{(n,m)}(S)$ forms a subsemigroup of S. ❑

Structurally inverse semigroups form a very important class of structurally orthodox semigroups. That, in itself, forms a new and a fertile area of research, especially in generalising known results about inverse semigroups. However, due to limited space and time, we will not study such semigroups. The proof of the following result follows easily from the characterisation of inverse semigroups, within the class of regular semigroups, by the uniqueness of inverses and by the commutativity of idempotent elements (compare the result with Theorem 0:2.5).

Corollary 3:4.2. *The following statements concerning a structurally regular semigroup S are equivalent:*

(i) $S/\theta(n,m)$ is an inverse semigroup.

(ii) S is structurally (n,m)-regular and idempotent $\theta(n,m)$-classes commute.

(iii) For any element a of S, if the elements a' and a'' both belong to $V_S(a;n,m)$ then a' and a'' are $\theta(n,m)$-related. ❑

Returning now to structurally orthodox semigroups, we have:

Proposition 3:4.3 *For any (n,m)-orthodox semigroup S, any (n,m)-idempotent e, and any (n,m)-inverse a' of a, both the elements a'ea and aea' are (n,m)-idempotents.*

Proof. Take any (n,m)-orthodox semigroup S, any (n,m)-inverse a'$\in V_S(a;n,m)$, and any e $\in E_{(n,m)}$. Then for all u in S^n and v in S^m we have that

$$u(a'ea)^2v = u(a'eaa'ea)v = u(a'eaa'e(aa'a))v = ua'(eaa')^2av$$

$$= ua'(eaa')av = u(a'ea)v;$$

and similarly, one can show also that the element aea' is also an (n,m)-idempotent. ❑

If the quotient $S/\theta(n,m)$ is an orthodox semigroup, then for each $e \in E_{(n,m)}$ consider the set:

$$E_{(n,m)}\langle e\rangle = \{x \in E_{(n,m)}: \ x\theta(n,m) \leq e\theta(n,m)\}$$

$$= \{x \in E_{(n,m)}: uexev = uev \text{ for all } u \in S^n, v \in S^m\}.$$

The ordering $\leq$ is of course the natural partial order on the band $E(S/\theta(n,m))$ defined by $f \leq e$ if and only if $ef = f = fe$. The following result shows that $E_{(n,m)}\langle e\rangle$ plays an analogous role to that played by the well known concept of *principal ideals* generated by idempotent elements within bands. We introduce the following notation: for any $a' \in V_S(a;n,m)$ let

$$\vartheta(E_{(n,m)}\langle a'a\rangle, a', E_{(n,m)}\langle aa'\rangle) = \{ x : (x,ea'f) \in \theta(n,m), \ \ e\in E\langle a'a\rangle , f\in E\langle aa'\rangle\}.$$

Proposition 3:4.4 *For any (n,m)-orthodox semigroup S and any (n,m)-inverse element a' of a, we have*

$$V_S(a;n,m) = \vartheta(E_{(n,m)}\langle a'a\rangle, a', E_{(n,m)}\langle aa'\rangle).$$

Proof. Let $e \in E_{(n,m)}\langle a'a\rangle$, $f \in E_{(n,m)}\langle aa'\rangle$, where $a' \in V_S(a;n,m)$. Then since

$u(a'a)e(a'a)v = ua'av$ and $u(aa')f(aa')v = u(aa')v$, we have that

$$ua(ea'f)av = u[aa'a(ea'aa'f)aa'a]v = u[a(a'aea'a)(a'faa')a]v$$

$$= u[a(a'a)(a'faa')a]v = u[aa'(aa'faa')a]v$$

$$= u[aa'(aa')a]v = u[aa'a]v = uav;$$

and also,

$$u[(ea'f)a(ea'f)]v = u[(ea'aa'f)aa'aa'a(ea'aa'f)]v$$

$$= u[ea'(aa'faa')a(a'aea'a)a'f]v$$

$$= u[ea'(aa')a(a'a)a'f]v$$

$$= u[ea'aa'f]v = uea'fv;$$

and hence $ea'f$ is an (n,m)-inverse of a. We have thus proved the containment

$$\vartheta(E_{(n,m)}\langle a'a\rangle, a', E_{(n,m)}\langle aa'\rangle) \subseteq V_S(a;n,m).$$

To prove the reverse containment, take any $a^* \in V_S(a;n,m)$. Then since

$$ua^*v = ua^*aa^*v = u(a^*a)a'(aa^*)v$$

we see that the elements a^* and $(a^*a)a'(aa^*)$ are $\theta(n,m)$-related in S. Now, we also observe that

$$u(a^*a)(a'a)(a^*a)v = u[a^*(aa'a)a^*a]v = u[a^*aa^*a]v = ua^*av;$$

and so $a^*a \in E_{(n,m)}\langle a'a\rangle$. Similarly, one can show that $aa^* \in E_{(n,m)}\langle aa'\rangle$. These, together, imply that $(a^*a)a'(aa^*)$ is contained in

$$\vartheta(E_{(n,m)}\langle a'a\rangle, a', E_{(n,m)}\langle aa'\rangle).$$

Since a^* is $\theta(n,m)$-related to the element $(a^*a)a'(aa^*)$, which belongs to the set $E_{(n,m)}\langle a'a\rangle\, a'E_{(n,m)}\langle aa'\rangle$, it follows that $a^* \in \vartheta(E_{(n,m)}\langle a'a\rangle, a', E_{(n,m)}\langle aa'\rangle)$, and thus the reverse containment holds. ❑

Theorem 3:4.5 *A (n,m)-regular semigroup S is (n,m)-orthodox if and only if for all elements a and b of S,*

$$V_S(a;n,m) \cap V_S(b;n,m) \neq \varnothing \text{ implies that } V_S(a;n,m) = V_S(b;n,m)\,.$$

Proof. Take any structurally (n,m)-orthodox semigroup, and suppose that the element $x \in V_S(a;n,m) \cap V_S(b;n,m) \neq \varnothing$. Then the elements a and b belong to $V_S(x;n,m)$, and so we have

$$E_{(n,m)}\langle xa\rangle = E_{(n,m)}\langle xb\rangle \qquad \text{and} \qquad E_{(n,m)}\langle ax\rangle = E_{(n,m)}\langle bx\rangle.$$

By Proposition 3:4.4 we have

$$\begin{aligned} V_S(a;n,m) &= \vartheta(E_{(n,m)}\langle xa\rangle, x, E_{(n,m)}\langle ax\rangle) \\ &= \vartheta(E_{(n,m)}\langle xb\rangle, x, E_{(n,m)}\langle bx\rangle) = V_S(b;n,m). \end{aligned}$$

Conversely, take any structurally (n,m)-regular semigroup S such that the condition in the theorem is satisfied. Take any (n,m)-idempotents e,f of $E_{(n,m)}$, and let x be an (n,m)-inverse of ef. That is, $u(ef)x(ef)v = uefv$ and $uxefxv = uxv$. Then the elements fxe and efxe belong to $E_{(n,m)}$ and the element fxe is contained in

$$V_S(fxe;n,m) \cap V_S(efxe;n,m).$$

By the hypothesis, this implies that $V_S(fxe;n,m) = V_S(efxe;n,m)$. Now, again it is straightforward to verify that ef belongs to $V_S(fxe;n,m)$; and hence ef belongs to $V_S(efxe;n,m)$ and so

$$uefv = u(ef)(efxe)(ef)v = uef(efxef)v = u(ef)^2v.$$

Thus it follows that S is structurally (n,m)-orthodox as required. ❑

Take any inverse semigroup congruence ρ on the orthodox semigroup $S/\theta(n,m)$, and consider the congruence on S given by

$$\omega = \{(a,b) \in S \times S : (a\theta(n,m), b\theta(n,m)) \in \rho,\ a,b \in S\}.$$

Then since $S/\omega \cong (S/\theta(n,m))/\rho$, it follows that ω is an inverse semigroup congruence on S. Therefore the number of inverse semigroup congruences on a structurally (n,m)-orthodox semigroups S is at least the number of inverse semigroup congruences on $S/\theta(n,m)$. The next result characterises the least inverse semigroup congruence on a structurally (n,m)-orthodox semigroup.

Theorem 3:4.6 *For any (n,m)-orthodox semigroup S the relation*

$$\gamma = \{(a,b): V_S(a;n,m) = V_S(b;n,m)\}$$

is the least inverse semigroup congruence on S.

Proof. Take any $(a,b) \in \gamma$ and any c in S. Then for any (n,m)-inverses c' of c and any a' of a, we have that a'c' belongs to

$$V_S(ca;n,m) \cap V_S(cb;n,m);$$

and so $V_S(ca;n,m) = V_S(cb;n,m)$ from Theorem 3:4.5, which implies that the elements ca and cb are γ-related. Similarly, $(ac,bc) \in \gamma$ and so the relation γ is a congruence. It is not difficult to see that S/γ is regular. To prove the commutativity of idempotent γ-classes, we observe that for any (n,m)-idempotent elements e and f, we have from Proposition 3:4.4 that

$$\begin{aligned} V_S(ef;n,m) &= E_{(n,m)}\langle ef\rangle \\ &= \{x \in E_{(n,m)} : x\theta(n,m) \le (ef)\theta(n,m) \text{ in } E(S/\theta(n,m)),\ e,f \in E_{(n,m)}\} \\ &= \{x \in E_{(n,m)} : x\theta(n,m) \le (fe)\theta(n,m) \text{ in } E(S/\theta(n,m)),\ e,f \in E_{(n,m)}\} \\ &= E_{(n,m)}\langle fe\rangle = V_S(fe;n,m). \end{aligned}$$

The third equality holds since in the band $E = E(S/\theta(n,m))$, $E\langle ef\rangle = E\langle fe\rangle$ for any $e,f \in E$ (For a proof of this, see (1.5) - (1.6) in Chapter VI of Howie (1976)). Hence, γ is an inverse semigroup congruence on S. Now, take any inverse semigroup congruence ρ on S, and any $(a,b) \in \gamma$. Then we have that $V_S(a;n,m) = V_S(b;n,m)$ and so for any x in $V_S(a;n,m)$ $(= V_S(b;n,m))$ both $a\rho$ and $b\rho$ are (n,m)-inverses of $x\rho$ in S/ρ. But since S/ρ is regular, the concepts of (n,m)-inverse and the usual (von Neumann) inverses coincide. Since the inverse semigroup S/ρ has unique inverses, we must have $a\rho = b\rho$. Hence $\gamma \subseteq \rho$. ❑

The least inverse semigroup congruence on an orthodox semigroup was determined by T.E. Hall (1969) to be

$$\gamma = \{(a,b) \in S \times S: V(a) = V(b)\}.$$

Yamada (1967) considered this congruence for generalised inverse semigroups.

A congruence ρ is defined to be a group congruence if S/ρ is a group. In Meakin (1972) the least group congruence on an orthodox semigroup was characterised in the following two ways:

$$\sigma = \{(a,b): V(fa) = V(fb) \text{ for some } f \in E(S)\}$$

and

$$\sigma = \{(a,b): eae = ebe \text{ for some } f \in E(S)\}.$$

Using an argument similar to those used earlier, to show that structurally orthodox semigroups have inverse semigroup congruences, one can also show that structurally orthodox and structurally inverse semigroups have group congruences.

Theorem 3:4.7 *For any structurally (n,m)-inverse semigroup S, the relation*

$$\sigma = \left\{ (x,y): (ex,ey) \in \theta(n,m) \textit{ for some } e \in E_{(n,m)}(S) \right\}$$

is the least group congruence on S.

Proof. The relation σ is clearly reflexive and symmetric. To show that it is transitive, take any $(x,y), (y,z) \in \sigma$. Then there exist some elements of $E_{(n,m)}(S)$, say e and f, such that for all u in S^n and v in S^m,

$$u(ex)v = u(ey)v \qquad \text{and} \qquad u(fy)v = y(fz)v.$$

Now,

$$u[(ef)x]v = u(fex)v = u(fey)v = u(efy)v = u(efz)v = u[(ef)z]v,$$

and since the element ef is an (n,m)-idempotent, it follows that $(x,z) \in \sigma$. Hence σ is an equivalence relation. Since $\theta(n,m)$ is a right congruence, so is the relation σ. To show that it is also a left congruence, take any $(x,y) \in \sigma$ and any z in S. Then there exists an (n,m)-idempotent e such $(ex,ey) \in \theta(n,m)$, and equivalently, for all u in S^n and v in S^m

$$u[(zez')zx]v = u[z(z'z)ex]v = u[z(z'z)ey]v = u[ze(z'z)y]v = u[(zez')zy]v.$$

Since zez' is (n,m)-idempotent (from Proposition 3:4.3), it follows that $(zx,zy) \in \sigma$; and hence σ is a congruence. For each x in S, it can be shown that the elements (xx')x and (xx')(xx'x) are $\theta(n,m)$-related, and so it follows that $(x, xx'x) \in \sigma$, proving that S/σ is regular. To show that it is a group, take any (n,m)-idempotents e and f of S. Then for all u in S^n and v in S^m we have

$$u(ef)ev = u(ee)fv = u(ef)fv,$$

and so $((ef)e,(ef)f) \in \theta(n,m)$, which implies that $(e,f) \in \sigma$ and so S has a unique idempotent σ-class. To show that σ is indeed the least group congruence on S, as claimed, take any group congruence, say ρ, and any σ-related elements x and y. Then there exists a (n,m)-idempotent element e such that for all u in S^n and v in S^m, $u(ex)v = u(ey)v$ by definition of σ. This implies that since ρ is a congruence

$$(u\rho)\,(e\rho)\,(x\rho)\,(v\rho) = (u\rho)\,(e\rho)\,(y\rho)\,(v\rho).$$

But since $e\rho$ is the unique identity element of the group S/ρ, we must have $(u\rho)(x\rho)(v\rho) = (u\rho)(y\rho)(v\rho)$, and this implies that the elements $x\rho$ and $y\rho$ are $\theta(n,m)$-related in the group S/ρ. But since groups are reductive, we must have $x\rho = y\rho$ and so $(x,y) \in \rho$, proving that $\sigma \subseteq \rho$. ❑

The next result follows easily in view of Lemma 3:3.20.

Corollary 3:4.8 *For any generalised inverse semigroup S, the relation*

$$\sigma = \{\ (a,b): xeay = xeby \textit{ for all } x,y \textit{ in } S, \textit{ and any idempotent } e\ \}$$

is the least group congruence. ❑

We say a semigroup is *reductive* if $\theta(i,j)$ is equal to the identity relation for every ordered pair (i,j) of non negative integers.

Corollary 3:4.8" *For any structurally (n,m)-orthodox semigroup S, the least group congruence on S is given by*

(i) $\sigma = \{\ (a,b) : V(ea;n,m) = V(eb;n,m) \textit{ for some } e \in E_{(n,m)}\ \}$

and also by

(ii) $\sigma = \{\ (a,b) : (eae,ebe) \in \theta(n,m) \textit{ for some } e\ E_{(n,m)}\ \}$

Proof. Suppose that $S/\theta(n,m)$ is orthodox. By the commutativity of the diagram in Theorem 1:1.4, it is easy to see that $\theta(n,m)$ is contained in any reductive semigroup congruence on S. In particular, if ω is a group congruence, then $\theta(n,m) \subseteq \omega$; and moreover, if ρ is the least group congruence on $S/\theta(n,m)$, then the following is the least group congruence on S:

$$\sigma = \{(a,b): (a\theta(n,m), b\theta(n,m)) \in \rho\}$$

and $S/\sigma \cong (S/\theta(n,m))/\rho$. We have (i) and (ii) by letting ρ take one of the two different expressions of the least group congruence on $S/\theta(n,m)$ determined by Meakin (as pointed out just before Theorem 3:4.7). ❑

A congruence δ will be called *idempotent $\theta(n,m)$-class separating* if

$$\delta \cap (E_{(n,m)} \times E_{(n,m)}) \subseteq \theta(n,m).$$

Every (structurally regular) semigroup has at least one idempotent $\theta(n,m)$-class separating congruence, namely the congruence $\theta(n,m)$ itself.

Theorem 3:4.9 *For any (n,m)-orthodox semigroup S, the relation given below is the maximum idempotent $\theta(n,m)$-class separating congruence*

$$\mu = \left\{ \begin{array}{l} (a,b): \forall e \in E_{(n,m)}(S), \exists a' \in V_S(a;n,m), \exists b' \in V_S(b;n,m), \textit{ such that both} \\ \qquad (a'ea,b'eb) \textit{ and } (aea',beb') \textit{ are } \theta(n,m)\textit{-related pairs in } S \end{array} \right\}$$

Proof. The relation μ is clearly reflexive and symmetric. To show that it is also transitive, take any (a,b), (c,d) ∈ μ. Then there exists a' in $V_S(a;n,m)$, b' and b* in $V_S(b;n,m)$, and c' in $V_S(c;n,m)$ such that for all elements e, u and v in $E_{(n,m)}$, S^n and S^m, respectively, we have

$$ua'eav = ub'ebv, \qquad uaea'v = ubeb'v$$

and

$$ub^*ebv = uc'ecv. \qquad ubeb^*v = ucec'v.$$

Next, we will prove that $å = b^*ba'bb' \in V_S(a;n,m)$.

Accordingly,

$$ubb'av = u[bb'(a(a'a)a')a]v = u[bb'(b(a'a)b')a]v = u[b(a'a)b')a]v$$
$$= u[a(a'a)a')a]v = uav$$

and also,

$$u[a(b^*b)a']v = u[b(b^*b)b']v = ubb'v.$$

We have in fact shown, by the above two sets of equalities, that (bb'a,a) and (abb*a',bb') are θ(n,m)-related pairs in S. These observations will be of use in the following:

Now,

$$uaåav = u[\ a(b^*b)a'(bb')av = u[bb'a]v = uav,$$

and

$$\begin{aligned} uåaåv &= u[b^*ba'bb'\ a(b^*b)a'bb']v \\ &= u[b^*ba'(bb'a)\ (b^*b)a'bb']v && \text{(by associativity)} \\ &= u[b^*ba'(\ a)\ (b^*b)a'bb']v && \text{(since } (bb'a,a) \in \theta(n,m)\text{)} \\ &= u[b^*ba'(\ ab^*ba')\ bb']v && \text{(by associativity)} \\ &= u[b^*ba'(bb')\ bb']v && \text{(since } (abb^*a',bb') \in \theta(n,m)\text{)} \\ &= u[b^*ba'(bb')^2]v \\ &= u[b^*ba'(bb')]v && \text{(since } bb' \text{ is } (n,m)\text{-idempotent)} \\ &= uåv. \end{aligned}$$

By a similar argument, one can show that $ç = b^*bcbb'$ belongs to $V_S(c;n,m)$. Now, for all e in $E_{(n,m)}$,

$$uåeav = u[b^*b\ a'(bb'e)a]v = u[b^*bb'(bb'e)b]v = u[b^*bb'eb]v$$

$$= u[b^*b\ b^*(bb'eb)]v = u[b^*b\ c^*(bb'e)c]v = uçecv,$$

and

$$uaeåv = u[a(eb^*b)a'bb']v = u[b(eb^*b)b'bb']v = u[b(eb^*b)b']v$$

$$= u[b(eb^*b)b^*bb']v = u[c(eb^*b)c^*bb']v = u[ceç]v;$$

and thus we have shown that $(a,c) \in \mu$ which proves the transitivity of μ. To show that μ is a right congruence as well, take any $(a,b) \in \mu$ and any $c \in S$ and any $c' \in V_S(c;n,m)$. Then for all $e \in E_{(n,m)}(S)$,

$$u[(ac)'e(ac)]v = u[c'(a'ea)c]v = u[c'(b'eb)c] = u[(bc)'e(bc)]v.$$

Also, since $cec' \in E_{(n,m)}$,

$$u[(ac)e(ac)']v = u[a(cec')a']v = u[b(cec)b']v = u[(bc)e(bc)']v.$$

We have thus established that $(ac,bc) \in \mu$ and hence μ is a right congruence. Similarly, $(ca,cb) \in \mu$. It is also easy to see that S/μ is regular. For any $(e,f) \in \mu \cap (E_{(n,m)} \times E_{(n,m)})$ there exists e' in $V_S(e;n,m)$ and f' in $V_S(f;n,m)$ such that

$$uf'efv = ue'eev = ue'ev,$$

and so

$$uefv = uee'efv = uef'effv = uef'efv = uee'eev = uev.$$

In the same way, one can also show that

$$uefe'v = ufff'v = uff'v$$

and hence

$$uefv = ueff'fv = ueefe'fv = uefe'fv = ufff'fv = ufv.$$

Therefore, $e\theta(n,m) = (ef)\theta(n,m) = f\theta(n,m)$; and thus $(e,f) \in \theta(n,m)$, proving that μ is an idempotent $\theta(n,m)$-class separating congruence. Finally, suppose that ρ is an idempotent $\theta(n,m)$-class separating congruence, and take any $(a,b) \in \rho$. Then since $a'\rho = (a\rho)' = (b\rho)' = b'\rho$, we have $(a',b') \in \rho$ for some a' in $V_S(a;n,m)$ and b' in $V_S(b;n,m)$. Then since ρ is a congruence, both $(a'ea,b'eb)$ and (aea',beb') are ρ-related elements for all $e \in E_{(n,m)}$. Then since the elements a'ea, b'eb, aea' and beb' are all (n,m)-idempotents, and since ρ is idempotent $\theta(n,m)$-class separating by assumption, we must have $(a'ea,b'eb), (aea',beb') \in \theta(n,m)$. This implies that $(a,b) \in \mu$ and hence $\rho \subseteq \mu$, proving that μ is the maximum idempotent $\theta(n,m)$-class separating congruence on S. ❑

In Howie (1964) the concept of *idempotent separating congruences* was introduced, and the maximum idempotent separating congruence on an inverse semigroup was found to be

$$\mu = \{(a,b): aea^{-1} = beb^{-1} \text{ for all } e \in E(S)\}.$$

A slightly different expression for μ was found by Lallement (1967). In Meakin (1972(a)) the maximum idempotent separating congruence on an orthodox semigroup was found to be as follows:

$$\mu = \{(a,b): aea' = beb' \text{ and } a'eb = b'eb\ ,\ b' \in V(b),\ a' \in V(a), \text{ and for all } e \in E(S)\}.$$

An analogous characterisation of this congruence was made for regular semigroups by Meakin (1972(b)). Although the idea of *idempotent $\theta(n,m)$-class separating congruence* is a generalisation of the concept of *idempotent-separating-congruences* (see Howie (1976)), it is a distinct concept since both these congruences do exist in orthodox semigroups but are different. We refer the reader to Theorem 1.17 of Chapter IV in Howie (1976) for the statement and proof of the maximum idempotent separating congruence on orthodox semigroups.

Corollary 3:4.10 *For any orthodox semigroup S, and for any $(i,j) \in \{0,1\}\times\{0,1\}$, the maximum idempotent $\theta(i,j)$-class separating congruence*

$$\mu = \left\{ (a,b)\text{: both } (a'ea,b'eb) \text{ and } (aea',beb') \text{ are } \theta(i,j)\text{-related pairs in } S \text{ for } a' \in V(a),\ b' \in V(b) \text{ and for all } e \in E(S) \right\}$$ ❑

Problem 3:4.11 *Determine the maximum idempotent separating congruence on structurally [regular, orthodox, inverse] semigroups?* ❑

Problem 3:4.12 *Determine the least group congruence on structurally regular semigroups, generalising Theorem 3:4.7?* ❑

Problem 3:4.13 *Find the maximum idempotent $\theta(n,m)$-class separating congruence on structurally regular semigroups, generalising Theorem 3:4.9?* ❑

Problem 3:4.14 *Is there a structure theorem for structurally orthodox semigroups which generalises T.E. Hall's structure theorem for orthodox semigroups (see exposition in Howie (1976))?* ❑

3:5 ORTHODOX RIGHT QUASI NORMAL BANDS OF GROUPS

In this final section of the chapter on structurally regular semigroups, we intend to demonstrate how regular semigroups could be studied using the family of congruences: $\{\theta(n,m): n \geq 0,\ m \geq 0\}$. Of course, every regular semigroup is structurally regular.

We will generalise the concept of strong semilattices to a concept of strong bands. As a special case of this, we present a structure theorem for orthodox right quasi normal bands of groups. This result unifies and generalises the characterisations of left normal bands as strong semilattices of left zero bands, due to Yamada, and Clifford semigroups as strong semilattices of groups (see Proposition 5.16 and Theorem 2.1 in Chapter IV of Howie (1976)).

The results presented in this chapter were first announced at an international semigroup conference held at Qingdao University, China, in 1993, and were later published in the *Southeast Asian Mathematical Society* (see Kopamu (1994)). We have made some notational changes here to be consistent with the rest of the thesis, although the main results have remained the same.

Let Γ be a band, of type $\mathcal{A}$, and $\{S_\alpha: \alpha \in \Gamma\}$ be a collection of pairwise disjoint semigroups of the same type C, indexed by Γ. Suppose that for each $\alpha \in \Gamma$ and $\beta \in \Gamma\alpha\Gamma = \{\gamma\alpha\delta : \gamma,\delta \in \Gamma\}$, there exists a homomorphism $\Phi_{\alpha,\beta}: S_\alpha \to S_\beta$, such that the following additional conditions are satisfied:

(3:5.1) for each $\alpha \in \Gamma$ and for any $a,b \in S_\alpha$, $(a\Phi_{\alpha,\alpha})(b\Phi_{\alpha,\alpha}) = ab$

(3:5.2) for any $\alpha,\beta,\gamma \in \Gamma$, if the maps $\Phi_{\alpha,\beta}$ and $\Phi_{\beta,\gamma}$ exist
(i.e $\beta \in \Gamma\alpha\Gamma$ and $\gamma \in \Gamma\beta\Gamma$)
then the map $\Phi_{\alpha,\gamma}$ also exists and is equal to the composition (from left to right) of $\Phi_{\alpha,\beta}$ and $\Phi_{\beta,\gamma}$
(i.e. $\Phi_{\alpha,\gamma} = \Phi_{\alpha,\beta}\Phi_{\beta,\gamma}$).

Lemma 3:5.3 *Take a collection* $\{S_\alpha: \alpha \in \Gamma\}$ *of semigroups satisfying the conditions* (3:5.1) *and* (3:5.2), *and define a binary operation* $\oplus$ *on the set* $S = \bigcup \{S_\alpha: \alpha \in \Gamma\}$ *by for any* $a \in S_\alpha$ *and* $b \in S_\beta$, $a \oplus b = (a\Phi_{\alpha,\alpha\beta})(b\Phi_{\beta,\alpha\beta})$.
Then $(S,\oplus)$ *is a semigroup.*

Proof. For any $\alpha,\beta,\gamma \in \Gamma$ we have $\alpha\beta = \alpha\alpha\beta \in \Gamma\alpha\Gamma$, $\alpha\beta = \alpha\beta\beta \in \Gamma\beta\Gamma$ and so the maps $\Phi_{\alpha,\alpha\beta}$ and $\Phi_{\beta,\alpha\beta}$ do exist. To prove the associativity of $\oplus$, take any $a \in S_\alpha$ $b \in S_\beta$, $c \in S_\gamma$ and consider:

$$[a\oplus b]\oplus c = [(a\Phi_{\alpha,\alpha\beta})(b\Phi_{\beta,\alpha\beta})]\oplus c$$

$$= [(a\Phi_{\alpha,\alpha\beta})(b\Phi_{\beta,\alpha\beta})]\Phi_{\alpha\beta,\alpha\beta\gamma}\,(c\Phi_{\gamma,\alpha\beta\gamma})$$

(the maps $\Phi_{\alpha\beta,\alpha\beta\gamma}$ and $\Phi_{\gamma,\alpha\beta\gamma}$ exist since $\alpha\beta\gamma\in\Gamma(\alpha\beta)\Gamma$ and $\alpha\beta\gamma\in\Gamma\gamma\Gamma$)

$$= [(a\Phi_{\alpha,\alpha\beta}\Phi_{\alpha\beta,\alpha\beta\gamma})(b\Phi_{\beta,\alpha\beta}\Phi_{\alpha\beta,\alpha\beta\gamma})]\,(c\Phi_{\gamma,\alpha\beta\gamma})$$

(since $\Phi_{\alpha\beta,\alpha\beta\gamma}$ is a homomorphism by assumption)

$$= [(a\Phi_{\alpha,\alpha\beta\gamma})(b\Phi_{\beta,\alpha\beta\gamma})]\,(c\Phi_{\gamma,\alpha\beta\gamma})$$

(by the transitivity condition (3:5.2))

$$= (a\Phi_{\alpha,\alpha\beta\gamma})[(b\Phi_{\beta,\alpha\beta\gamma})\,(c\Phi_{\gamma,\alpha\beta\gamma})]$$

(by the associativity of the product on $S_{\alpha\beta\gamma}$)

$$= (a\Phi_{\alpha,\alpha\beta\gamma})[(b\Phi_{\beta,\beta\gamma}\Phi_{\beta\gamma,\alpha\beta\gamma})\,(c\Phi_{\gamma,\beta\gamma}\Phi_{\beta\gamma,\alpha\beta\gamma})]$$

(again, by the transitivity condition (3:5.2)))

$$= (a\Phi_{\alpha,\alpha\beta\gamma})[(b\Phi_{\beta,\beta\gamma})\,(c\Phi_{\gamma,\beta\gamma})]\Phi_{\beta\gamma,\alpha\beta\gamma}$$

(since $\Phi_{\beta\gamma,\alpha\beta\gamma}$ is a homomorphism by assumption)

$$= (a\Phi_{\alpha,\alpha\beta\gamma})[b\oplus c]\Phi_{\beta\gamma,\alpha\beta\gamma}$$

$$= a\oplus[b\oplus c] \qquad \square$$

We have from (3:5.1) that the multiplication on each S_α is preserved by the binary operation $\oplus$. We shall call $(S,\oplus)$ a *strong $\mathcal{A}$-band of $\mathcal{C}$-semigroups*. In particular, if Γ is a semilattice, then $(S,\oplus)$ is a *strong semilattice* of $\mathcal{C}$-semigroups in the sense of Howie (1976). We shall denote the semigroup so constructed in the following way:

(3:5.4) $\qquad (S,\oplus) = (\Gamma;\{S_\alpha\colon \alpha\in\Gamma\}\,;\{\phi_{\alpha,\beta}\colon\ \alpha\in\Gamma,\beta\in\Gamma\alpha\Gamma\})$.

The following result characterises certain classes of right quasi normal bands (see Figure 0:4.13) within the class of regular semigroups .

Lemma 3:5.5 *Let* B *be a regular semigroup. Then the following statements hold:*

(a) B *is a right quasi normal band if and only if* $B/\theta(1,0)$ *is a right regular band.*

(b) B *is a normal band if and only if* $B/\theta(1,0)$ *is a right normal band.*

(c) B *is a left zero band if and only if* $B/\theta(1,0)$ *is trivial.*

(d) B *is a rectangular band if and only if* $B/\theta(1,0)$ *is a right zero band.*

(e) B *is a left normal band if and only if* $B/\theta(1,0)$ *is a semilattice*

Proof. Take a regular semigroup B, and suppose that $B/\theta(1,0)$ is a band. The conclusion that B is a band is due to the following: since B is a union of idempotent $\theta(1,0)$-classes, we have that for each element x of B, the elements x and xx are $\theta(1,0)$-related in B. Therefore, for any inverse x' of x, we have

$$x = xx'x = (xx')xx = (xx'x)x = xx;$$

and so B is a band. Now, suppose that $B/\theta(1,0)$ is a right quasi normal band. Then from Figure 0:4.13, $B/\theta(1,0)$ satisfies $xy = yxy$. By Theorem 2:2.6, B satisfies the identity $zxy = zyxy$ and hence B is a right quasi normal band (see Figure 0:4.13). Conversely, suppose that B is right quasi normal. Then clearly $B/\theta(1,0)$ is a band since it is a homomorphic image. Moreover, for any elements a and b of B, and for all $x \in B$, since $xab = xbab$, it follows by Theorem 2:2.6 that the band $B/\theta(1,0)$ satisfies the identity $ab = bab$, and hence the quotient $B/\theta(1,0)$ is a right regular band. This proves (a). The remaining statements (b) - (e) can be proved similarly. ❑

We also have the following dual statements of Lemma 3:5.5.

Lemma 3:5.6 *Let* B *be a regular semigroup. Then the following statements hold:*

(a) B *is a left quasi normal band if and only if* $B/\theta(0,1)$ *is a left regular band.*

(b) B *is a normal band if and only if* $B/\theta(0,1)$ *is a left normal band.*

(c) B *is a right zero band if and only if* $B/\theta(0,1)$ *is trivial.*

(d) B *is a rectangular band if and only if* $B/\theta(0,1)$ *is a left zero band*

(e) B *is a right normal band if and only if* $B/\theta(0,1)$ *is a semilattice* ❑

Combining the results in Lemma 3:5.5 and Lemma 3:5.6 we have:

Corollary 3:5.7 *A regular semigroup B is a [normal, rectangular] band if and only if* $B/\theta(1,1)$ is a *[semilattice, trivial] band.* ❑

Theorem 3:5.8 *A semigroup* S *is a strong [right,left] regular band of [left,right] zero bands if and only if it is a [right,left] quasi normal band.*

Proof. Let S be a right quasi normal band, and denote by $\{S_\alpha: \alpha \in \Gamma\}$ the family of all $\theta(1,0)$-classes of S; and for each $\alpha \in \Gamma$ and $\beta \in \Gamma\alpha\Gamma$, one can define the map $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$, $x\phi_{\alpha,\beta} = xb$, for any b in S_β. We see that the map is well defined since S_β is a $\theta(1,0)$-class of S. To prove that S is a strong right regular band of left zero bands, we will show that each S_α is a left zero band, that conditions (3:5.1) and (3:5.2) are satisfied, and that the product on S can be expressed in terms of these maps and $\{S_\alpha: \alpha \in \Gamma\}$ as in Lemma 3:5.3. Accordingly, for any elements x and y in a $\theta(1,0)$-class, say S_α, we have $xy = xx = x$. Therefore each member of $\{S_\alpha: \alpha \in \Gamma\}$ is a left zero band and hence a subsemigroup of S.

Next, we observe from Lemma 3:5.5 (a) that the indexing semigroup $\Gamma = S/\theta(1,0)$ is a right regular band. For any $\beta \in \Gamma\alpha\Gamma$, there exist $\delta, \gamma \in \Gamma$ such that $\beta = \delta\alpha\gamma$. Since Γ is right regular, we see that α is a left identity of β since:

$$\beta = \delta\alpha\gamma = \alpha(\delta\alpha\gamma) = \alpha\beta$$

and so $\phi_{\alpha,\beta} = \phi_{\alpha,\alpha\beta}$. We have thus shown that the map $\phi_{\alpha,\beta}$ defined in (1:2.4) can be expressed alternatively as $\phi_{\alpha,\alpha\beta}$ for right quasi normal bands. Hence $S_{\alpha\beta} = S_\beta$. Next we see that each $\phi_{\alpha,\beta}$, where $\alpha \in \Gamma$ and $\beta \in \Gamma\alpha\Gamma$, is a homomorphism from S_α into S_β since for any elements $x,y \in S_\alpha$,

$$(x\phi_{\alpha,\beta})\,(y\phi_{\alpha,\beta}) = (x\phi_{\alpha,\beta}) = (xy)\phi_{\alpha,\alpha\beta}.$$

We have the first equality since S_β is a left zero band, and the second equality holds since S_α is a left zero band. Moreover, for any $(\alpha,\beta) \in \Gamma \times \Gamma$, the following equalities hold since Γ satisfies the identity $xy = yxy$:

$$\phi_{\alpha,\beta\alpha\beta} = \phi_{\alpha,\alpha\beta} \qquad \text{and} \qquad \phi_{\beta,\beta\alpha\beta} = \phi_{\beta,\alpha\beta}\,.$$

Hence this pair of homomorphisms maps the left zero bands S_α and S_β respectively into the left zero band $S_{\alpha\beta}$. We see that condition (3:5.1) holds since for any $x,y \in S_\alpha$

$$(x\phi_{\alpha,\alpha})\,(y\phi_{\alpha,\alpha}) = (x\phi_{\alpha,\alpha}) = xy.$$

We have the first equality since S_α is a left zero band, and the second equality from the definition of $\phi_{\alpha,\beta}$. By the equality $\phi_{\alpha,\beta} = \phi_{\alpha,\alpha\beta}$, and in view of Theorem 1:2.6, the transitivity condition (3:5.2) holds, namely

$$\phi_{\alpha,\beta}\,\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$$

for any $\alpha \in \Gamma$, $\beta \in \Gamma\alpha\Gamma$, and $\gamma \in \Gamma\beta\Gamma$. Finally, to prove that S is a strong right quasi normal band of the left zero bands, we will show that the multiplication on S can be expressed as in Lemma 3:5.3. Take any elements a and b in S_α and S_β, respectively. Then

$$(a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) = (a\phi_{\alpha,\alpha\beta}) = a(ab) = ab.$$

We have the first equality since $S_{\alpha\beta}$ is a left zero band, the second equality by definition of the map $\phi_{\alpha,\alpha\beta}$, and the third equality holds since $aa = a$. Thus we have proved that S is a strong right regular band of left zero bands. The proof of the converse is easy and is therefore omitted. The alternative statement holds by duality. ❑

Theorem 3:5.9 *A semigroup S is a strong [right normal, semilattice, right zero, trivial band] of left zero semigroups if and only if S is a [normal, left normal, rectangular, left zero] band.*

Proof. The proof follows much the same way as in the proof of Theorem 3:5.8, but where Lemma 3:5.5 (a) comes into play we use Lemma 3:5.5 (b), (c), (d) or (e) where appropriate. ❑

On any orthodox right quasi normal band of group S say, there exists a congruence γ (namely $\gamma = \mathcal{H}$, the Green's relation) such that S/γ is a right quasi normal band, and each γ-class is a group. Since γ is idempotent-separating, the band E(S) of all idempotents of S is isomorphic to S/γ. Therefore, for each idempotent e of S, there exists a group G_e (namely H_e) such that $S = \bigcup\{G_e: e \in E(S)\}$. Since E(S) is right quasi normal, the decomposition of E(S) given below is a strong right regular band of left zero bands (from Lemma 3:5.8):

$$E(S) = (\Gamma ; \{L_\alpha: \alpha \in \Gamma\} ; \{\phi_{\alpha,\alpha\beta}: (\alpha,\beta) \in \Gamma \times \Gamma\}).$$

Now, for each $\alpha \in \Gamma = E(S)/\theta(1,0)$, define

$$S_\alpha = \bigcup\{G_e: e \in L_\alpha\},$$

the union of all maximal subgroups of S such that the idempotent elements form the $\theta(1,0)$-class L_α of the band E(S). Then each S_α is a union of groups, closed under multiplication. Since the idempotents in S_α are $\theta(1,0)$-related, $E(S_\alpha)$ forms a left zero band and hence S_α is a left group. Since we are concerned with completely regular semigroups in this section, we shall denote by a^{-1} the unique inverse of a in the maximal subgroup containing a.

Lemma 3:5.10 *Let S and* $\{L_\alpha: \alpha \in \Gamma\}$ *be as described in the preceding discussion. For each* $\alpha \in \Gamma$, *and any* $\beta \in \Gamma\alpha\Gamma$, *the map defined below is a homomorphism of left groups:*

$$\vartheta_{\alpha,\beta}: S_\alpha \to S_\beta, \quad a \mapsto ae_z$$

where e_z *is any idempotent in the left group* S_β.

Proof. For any idempotents e and f in S_β, since e and f are $\theta(1,0)$-related in E(S), we have that for all x in S_α, $xe = xx^{-1}xe = xx^{-1}xf = xf$. Hence e and f are $\theta(1,0)$-related not just in E(S) but in the whole semigroup S, and so the map $\vartheta_{\alpha,\beta}$ is well defined. To see that it is a homomorphism, take any $\alpha \in \Gamma$ and

$\beta \in \Gamma\alpha\Gamma$, any $a \in G_x$ and $b \in G_y$ such that both G_x and G_y are subgroups of S_α. Then for any idempotent element e_z in S_β,

$$(ab)\vartheta_{\alpha,\beta} = (ab)e_z = a(be_z)$$

$$= a\,(b^{-1}be_z)\,(be_z)$$

(since $b^{-1}be_z$ is the identity element of the subgroup containing be_z)

$$= (aa^{-1}a)e_z\,(be_z) \qquad (\text{since } (a^{-1}a, b^{-1}b) \in \theta^S(1,0))$$

$$= (ae_z)(be_z)$$

$$= (a\vartheta_{\alpha,\beta})\,(b\vartheta_{\alpha,\beta})$$

For any $\alpha \in \Gamma$ and any $\beta \in \Gamma\alpha\Gamma$ there exists $\delta,\gamma \in \Gamma$ such that $\beta = \delta\alpha\gamma$. To show that for every $a \in S_\alpha$, the image $a\vartheta_{\alpha,\beta}$ is really contained in S_β, we see that for any idempotent $e_z \in S_\beta$,

$$a\vartheta_{\alpha,\beta} = ae_z \in S_{\alpha\beta} = S_{\alpha\delta\alpha\gamma} = S_{\delta\alpha\gamma} = S_\beta$$

The last two equalities hold since Γ is a right regular band and $\alpha\delta\alpha\gamma = \delta\alpha\gamma$; and that completes the proof. ❑

Theorem 3:5.11 *A semigroup S is a strong [right,left] regular band of [left,right] groups if and only if it is an orthodox [right,left] quasi normal band of groups.*

Proof. As in the proof of Theorem 3:5.8 for right quasi normal bands, orthodox right quasi normal bands of groups have the property that for any $(\alpha,\beta) \in \Gamma \times \Gamma$, the following pair of homomorphisms, namely $\vartheta_{\alpha,\alpha\beta}$ and $\vartheta_{\beta,\alpha\beta}$, (defined in Lemma 3:5.10) map the left groups S_α and S_β, respectively, into $S_{\alpha\beta}$. Then for any $a \in G_x \subseteq S_\alpha$ and $b \in G_y \subseteq S_\beta$, where G_y and G_x are maximal subgroups,

$$(a\vartheta_{\alpha,\alpha\beta})(b\vartheta_{\beta,\alpha\beta}) = (a\vartheta_{\alpha,\alpha\beta})(b\vartheta_{\beta,\beta\alpha\beta})$$

$$= (a(a^{-1}ab^{-1}b))\,[b(b^{-1}ba^{-1}ab^{-1}b)]$$

$$= (ab)(a^{-1}ab^{-1}b) = ab.$$

We have the last equality since $a^{-1}ab^{-1}b$ is the identity element of the group containing ab. We also observe that the indexing semigroup $\Gamma = E(S)/\theta(1,0)$ is a right regular band since $E(S)$ is a right quasi normal band (from Lemma 3:5.5 (a)). To show that S is a strong right regular band of left groups, we need only show that conditions (3.5.1) and (3:5.2) are satisfied. Now, condition (3:5.1) holds since for any $a,b \in S_\alpha$ we have

$$(a\vartheta_{\alpha,\alpha})(b\vartheta_{\alpha,\alpha}) = (aa^{-1}a)\,(bb^{-1}b) = ab \qquad (\text{since } aa^{-1}, bb^{-1} \in E(S_\alpha)).$$

The transitivity condition (3:5.2) holds since for any $\beta \in \Gamma\alpha\Gamma$ and $\gamma \in \Gamma\beta\Gamma$, and for any

$$a \in G_x \subseteq S_\alpha, \qquad b \in G_y \subseteq S_\beta, \text{ and } \quad c \in G_z \subseteq S_\gamma$$

we have

$$(a\vartheta_{\alpha,\beta})\vartheta_{\beta,\gamma} = (ab^{-1}b)\vartheta_{\beta,\gamma} = (ab^{-1}b)\, c^{-1}c = a(b^{-1}bc^{-1}c) = a\vartheta_{\alpha,\gamma}.$$

The last equality holds since the idempotent $b^{-1}bc^{-1}c = b^{-1}b\vartheta_{\beta,\gamma} \in S_\gamma$. Thus we have shown that S is a strong right regular band of the left groups: $\{S_\alpha: \alpha \in \Lambda\}$.

Conversely, take a strong right regular band of left groups say S,

$$S = (\Gamma\, ; \{S_\alpha: \alpha \in \Gamma\}\, ; \{\vartheta_{\alpha,\beta}: \alpha \in \Gamma\, , \beta \in \Gamma\alpha\Gamma\}\,).$$

Then S is clearly an orthodox band of groups. Since homomorphic images of idempotents are idempotents, the band E(S) of idempotents is a strong right regular band of left zero bands:

$$E(S) = (\Gamma\, ; \{E(S_\alpha): \alpha \in \Gamma\}\, ; \{\varphi_{\alpha,\beta}: \alpha \in \Gamma\, , \beta \in \Gamma\alpha\Gamma\}\,),$$

where each $E(S_\alpha)$ denotes the idempotent elements of S_α, and each $\varphi_{\alpha,\beta}$ is the restriction of $\vartheta_{\alpha,\beta}$ to the idempotent elements. We have from Theorem 3:5.8 that E(S) is a right quasi normal band, and it follows that S is an orthodox right quasi normal band of groups. Again, the alternative statement holds by duality. ❑

The following varieties of right quasi normal bands consist of semigroups which are not left reductive: left normal, normal, left zero and rectangular bands. The corollary below concerns orthodox bands of groups whose band of idempotents E(S) is from a variety of right quasi normal bands that is formed entirely by semigroups which are not left reductive.

Corollary 3:5.12 *A semigroup S is an orthodox [left normal, normal, left zero, rectangular] band of groups if and only if S is a strong [semilattice , right normal band, trivial, right zero band] of left groups.*

Proof. By replacing right regular band, wherever it occurs in the proofs of Lemma 3:5.10 and Theorem 3:5.11, by [semilattices, right normal, trivial or right zero] band, wherever appropriate, one can obtain a proof for each of these statements. ❑

In contrast to the orthodox semigroups considered in Corollary 3:5.12, we now consider in Corollary 3:5.13 those orthodox bands of groups, whose idempotent elements E(S) come from a variety of bands consisting entirely of left reductive semigroups. In fact, the following varieties of right quasi normal bands which are left reductive: right regular, right normal, semilattices, right zero and the trivial band.

Corollary 3:5.13 *A semigroup S is an orthodox [right regular, right normal, semilattice, right zero, trivial] band of groups if and only if S is a strong [right regular, right normal, semilattice, right zero, trivial] band of groups.* ❑

A *rectangular group* is an orthodox band of groups whose idempotents form a rectangular band. Alternatively, such a semigroup is a direct product of a rectangular band and a group.

Theorem 3:5.14 (Petrich (1973)) *A semigroup S is an orthodox normal band of groups if and only if it is a strong semilattice of rectangular groups.*

Proof. Take an orthodox normal band of groups $S = \bigcup\{G_e : e \in E(S)\}$ say, where G_e is the maximal subgroup containing the idempotent e. Define a relation δ on E(S) as follows:

(3:5.15) $\qquad \delta = \{(a,b) \in E(S) \times E(S) : xay = xby \text{ for all } x,y \in E(S)\}.$

Note that δ is just the congruence $\theta(1,1)$ on E(S). For any $(e,f) \in \delta$, we have $efe = eee = e$, and so every δ-class is a rectangular band. We see that $E(S)/\delta = \Gamma$ is a semilattice since for any $(e,f) \in \delta$, $xefy = xfey$. (In fact $\delta = \mathcal{D}(E(S))$ the Green's relation). Denote by $\{B_\alpha : \alpha \in \Gamma\}$ the family of all δ-classes. For each $\alpha \in \Gamma$ define

$$S_\alpha = \bigcup \{G_e : e \in B_\alpha\},$$

the union of all maximal subgroups of S such that the idempotent elements in them form the $\theta(1,1)$-class B_α of the band E(S). We see that each S_α forms a subsemigroup of S, and since $E(S_\alpha) = B_\alpha$ is a rectangular band, S_α is a rectangular group. For each $\alpha \in \Gamma$ and any $\beta \le \alpha$, define a map from S_α into S_β as follows:

$$\Phi_{\alpha,\beta} : S_\alpha \to S_\beta, \quad x \mapsto xx^{-1}fxfx^{-1}, \text{ for any } f \in E(S_\beta).$$

We see that this map is well defined: since any two idempotent elements e and f in S_β are δ-related, we have that for all x in S_α,

$$\begin{aligned} xx^{-1}fxfx^{-1} &= (xx^{-1}fxx^{-1})x(xx^{-1}fxx^{-1}) \\ &= (xx^{-1}exx^{-1})x(xx^{-1}exx^{-1}) = xx^{-1}exexx^{-1} \end{aligned}$$

We show next that this map is a homomorphism: take any $a,c \in S_\alpha$ and any $e \in E(S_\beta)$.
Then

$$(ac)\Phi_{\alpha,\beta} = (ac)(ac)^{-1}\, eace(ac)(ac)^{-1}$$

$$= (aa^{-1}cc)^{-1}\, eace(aa^{-1})(cc^{-1}) \quad (\text{since } (ac)^{-1}(ac) = aa^{-1}cc^{-1})$$

$$= (aa^{-1}aa)^{-1}\, eace(cc^{-1})(cc^{-1}) \quad (\text{since } (aa^{-1}, cc^{-1}) \in \delta)$$

$$= aa^{-1}(eace)(cc^{-1})$$

$$= aa^{-1}(ea)(eaa^{-1})\,(cc^{-1}e)(ce)\,(cc^{-1})$$

(since eaa^{-1} and $cc^{-1}e$ are identity elements of the maximal groups containing ea and ce, respectively)

$$= (aa^{-1}eaeaa^{-1})\,(cc^{-1}ececc^{-1})$$

$$= (a\Phi_{\alpha,\beta})\,(c\Phi_{\alpha,\beta})$$

We point out that since Γ is a semilattice, $\alpha\beta = \beta\alpha$ for any $\alpha,\beta \in \Gamma$ and so the homomorphisms $\Phi_{\beta,\beta\alpha}$ and $\Phi_{\alpha,\alpha\beta}$ map the rectangular groups S_α and S_β, respectively, into the same rectangular group $S_{\beta\alpha} = S_{\alpha\beta}$.

Next we will show that conditions (3:5.1) and (3:5.2) are both satisfied. Condition (3:5.1) holds since for any $a,b \in S_\alpha$ and $\alpha \in \Gamma$,

$$(a\Phi_{\alpha,\alpha})\,(b\Phi_{\alpha,\alpha}) = [aa^{-1}\,(aa^{-1})\,a\,(aa^{-1})\,aa^{-1}]\;\; [bb^{-1}\,(bb^{-1})\,b\,(bb^{-1})\,bb^{-1}]$$

$$= [aa^{-1}\,a\,aa^{-1}]\,[bb^{-1}\,b\,bb^{-1}] = ab.$$

Now, to show that the transitivity condition (3:5.2) is also satisfied, take any $\alpha \geq \beta \geq \gamma$ and any idempotent elements $e \in E(S_\beta)$ and $f \in E(S_\gamma)$. Then for any $a \in S_\alpha$, we have that

$$(a\Phi_{\alpha,\beta})\Phi_{\beta,\gamma} = gg^{-1}fgfgg^{-1}$$

(where $g = aa^{-1}eaeaa^{-1} = a\Phi_{\alpha,\beta}$ and $g^{1} = aa^{-1}ea^{-1}eaa^{-1}$)

$$= ((aa^{-1})(eaea^{-1}))\;(\,fgf)\;((aa^{-1})(eaea^{-1}))$$

(since $gg^{-1} = (aa^{-1}eaeaa^{-1})(aa^{-1}ea^{-1}eaa^{-1}) = (aa^{-1})(eaea^{-1})$)

$$= ((aa^{-1})(eaea^{-1}))\;(f(aa^{-1}eaeaa^{-1})f)\;((aa^{-1})(eaea^{-1}))$$

$$= (aa^{-1})eaea^{-1}faa^{-1})a(eaa^{-1}faa^{-1}eaea^{-1})$$

$$= ((aa^{-1})\,(h\;a\;k)$$

(where $h = eaea^{-1}\,faa^{-1}$ and $k = eaa^{-1}faa^{-1}eaea^{-1}$)

$$= (aa^{-1})\,(h\;a\;h)\,(aa^{-1})$$

(since $k = h(aa^{-1})$, which we verify below)

$$= a\Phi_{\alpha,\gamma} \qquad\qquad (\text{since } h \in E(S_\gamma));$$

The second last equality holds since $k = h(aa^{-1})$, which we verify as follows:

$$k = eaa^{-1}faa^{-1}eaea^{-1} = e(aa^{-1})^2 feaea^{-1}(aa^{-1}) = eaa^{-1}\,fe\,(aea^{-1})\,(aa^{-1})$$

$$= e(aea^{-1})aa^{-1}\,fe\,(aa^{-1}) = (eaea^{-1})\,fe\,(aa^{-1}) = eaea^{-1}f\,(aa^{-1}) = h\,(aa^{-1}).$$

Thus we have shown that (3:5.2) also holds. It remains to show that the multiplication on S can be expressed as in Lemma 3:5.3. For any element a in S_α and b in S_β we have

$$(a\Phi_{\alpha,\alpha\beta})\,(b\Phi_{\beta,\alpha\beta}) = (aa^{-1}bb^{-1})a(bb^{-1}aa^{-1})\,(bb^{-1}aa^{-1})b(aa^{-1}bb^{-1})$$

$$= (aa^{-1}bb^{-1})a(aa^{-1}aa^{-1})(bb^{-1}bb^{-1})b(aa^{-1}bb^{-1})$$

$$= (aa^{-1}bb^{-1})(ab)(aa^{-1}bb^{-1}) = ab.$$

The last equality holds since $aa^{-1}bb^{-1}$ is the identity of the group containing ab. Thus we have proved that S is a strong semilattice of the family $\{S_\alpha : \alpha \in \Gamma\}$ of rectangular groups. The converse is easy to prove. ❑

Corollary 3:5.16 (Kimura and Yamada (1958)) *A semigroup S is a normal band if and only if it is a strong semilattice of rectangular bands.*

Proof. In this particular case, $S = E(S)$ and so each S_α is just a δ-class, and for each $\alpha \geq \beta$ the definition of the map $\Phi_{\alpha,\beta}$ reduces simply to $x \mapsto xfx$, for any $f \in S_\beta$. ❑

Next, we present some results involving orthodox semigroups which are not necessarily bands of groups. These results seem quite trivial and easy to prove using our approach, but are not so easily recognisable otherwise. We shall call a regular semigroup a *right quasi normal banded semigroup* if the set E(S) of idempotents is a right quasi normal band. The following result characterises such semigroups within the class of all regular semigroups. A regular semigroup is called *$\mathcal{L}$-unipotent* if E(S) is a right regular band. Compare the following result with Lemma 3:5.5.

Theorem 3:5.17 *Let* S *be a regular semigroup. Then the following statements hold:*

(a) *S is a right quasi normal banded semigroup if and only if* $S/\theta(1,0)$ *is $\mathcal{L}$-unipotent.*

(b) S *is a generalised inverse semigroup if and only if* $S/\theta(1,0)$ *is a right generalised inverse semigroup.*

(c) S *is a left group if and only if* $S/\theta(1,0)$ *is a group.*

(d) S *is a rectangular group if and only if* $S/\theta(1,0)$ *is right group.*

(e) S *is a left generalised inverse semigroup if and only if* $S/\theta(1,0)$ *is an inverse semigroup.*

Proof. Suppose that S is regular. Suppose also that $\Gamma = S/\theta(1,0)$ is $\mathcal{L}$-unipotent. Then by definition, $E(\Gamma)$ is a right regular band. It follows by Lemma 3:5.5(a) that E(S) is a right quasi normal band. Hence, by definition, S is a right quasi normal banded semigroup. Conversely, if S is right quasi normal banded, then E(S) is right quasi normal. Then, again, by Lemma 3:3.5(a), $\Gamma = S/\theta(1,0)$ is a $\mathcal{L}$-unipotent semigroup. This proves (a). The remaining statements (b) - (e) can be proved similarly. ❑

The next result is the dual of Theorem 3:5.17.

Theorem 3:5.18 *Let* S *be a regular semigroup. Then the following statements hold:*

(a) S *is a left quasi normal banded semigroup if and only if* $S/\theta(0,1)$ *is* $\mathcal{R}$*-unipotent.*

(b) S *is a generalised inverse semigroup if and only if* $S/\theta(0,1)$ *is a left generalised inverse semigroup.*

(c) S *is a right group if and only if* $S/\theta(0,1)$ *is a group.*

(d) S *is a rectangular group if and only if* $S/\theta(0,1)$ *is a left group.*

(e) S *is a right generalised inverse semigroup if and only if* $S/\theta(0,1)$ *is an inverse semigroup.* ❑

By combining Theorem 3:5.17 and Theorem 3:5.18, we have:

Corollary 3:5.19 *Let* S *be a regular semigroup. Then the following statements hold:*

(a) S *is a generalised inverse semigroup if and only if* $S/\theta(1,1)$ *is an inverse semigroup.*

(b) S *is a rectangular group if and only if* $S/\theta(1,1)$ *is a group.* ❑

For any regular semigroup S, the congruences $\theta(1,0)$ and $\theta(0,1)$ are the least left reductive and the least right reductive congruences on S, respectively. We see from the proof of Theorem 3:1.6 that $\delta_1 = \theta(1,0) \cap \theta(0,1)$ is regular element separating congruence. The next result follows.

Corollary 3:5.20 *Every regular semigroup S is a spined product of its maximum left reductive homomorphic image, (namely, $S/\theta(1,0)$) and its maximum right reductive homomorphic image (namely, $S/\theta(0,1)$).*

Proof. Since $S^2 = S$ for regular semigroups, we have from Corollary 1:1.6, that the congruences $\theta(1,0)$ and $\theta(0,1)$ are the least left reductive and the least right reductive congruences, respectively. The map $a \mapsto (\theta(1,0),\theta(0,1))$ embeds S into the direct product $S/\theta(1,0) \times S/\theta(0,1)$. The kernel of this map is of course the congruence δ_1; and it turns out to be the identity relation for regular semigroups (see the proof of Theorem 3:1.6). ❑

Corollary 3:5.21 (Yamada (1967)) *Every generalised inverse semigroup S is a spined product of its maximum left and right generalised inverse semigroup homomorphic images.* ❑

Corollary 3:5.22 (Kimura and Yamada (1958)) *Every normal band B is a spined product of its maximum left normal band and its maximum right normal band homomorphic images.* ❑

CHAPTER 4

VARIETIES OF STRUCTURALLY GROUP SEMIGROUPS

In the remaining chapters of this thesis we will be concerned with varieties of semigroups. In the present chapter we introduce two new products on the lattice of all semigroup varieties. The first product (denoted here by $\oplus_{(n,m)}$) resembles the well known Mal'tsev product; and the second (and a more specialised) product $\otimes_{(n,m)}$ is defined on the lattice of all semigroup varieties formed by nilpotent extensions of rectangular groups. These products are in general distinct, but for certain special cases they do coincide; and in general the classes of semigroups so produced are not varieties. However, it is shown that every variety $\mathcal{V}$ formed entirely by nilpotent extensions of rectangular groups can be expressed uniquely as a product $\mathcal{V}= \mathcal{U}\otimes_{(n,m)}\mathcal{W}= \mathcal{U}\oplus_{(n,m)}\mathcal{W}$, where $\mathcal{U}$ is a variety formed by nilpotent extensions of rectangular bands, and $\mathcal{W}$ is a variety consisting entirely of groups.

It is shown in Lemma 4:3.5 that the lattice maps introduced in Theorem 2:2.6 do not map lattice intervals onto lattice intervals in general, although they are one-to-one and class intersection preserving. But whenever these maps are restricted to the subvarieties of a variety $\mathcal{V}$ consisting entirely of groups, they map intervals onto intervals (see Corollary 4:4.14); and hence they preserve varietal joins. Moreover, if $\mathcal{W}$ is a semigroup variety consisting entirely of groups, then $\mathcal{W}^{(n,m)}$ is shown to be the join of the subvarieties $\mathcal{T}^{(n,m)}$ and $\mathcal{W}$. This nice property does not hold in general, even when $\mathcal{W}$ is a semigroup variety consisting entirely of inverse semigroups.

A semigroup S is called a *nilpotent-extension* of T if there exists a positive integer $n\geq 1$ such that $S^n = T$. A semigroup S is said to be *n-nilpotent* if the ideal S^n is trivial, and the variety of all such semigroups is denoted by $\mathcal{N}_n$. Any semigroup from the class $\mathcal{N}= \bigcup \{\mathcal{N}_n: n=1,2,3, \dots \}$ is said to be *nilpotent.*

There do exist structurally trivial varieties $\mathcal{U}\neq \mathcal{V}$ and a variety $\mathcal{W}$ consisting of groups such that $\mathcal{U}\vee \mathcal{W}= \mathcal{V}\vee \mathcal{W}$. In this chapter we introduce the concept of a *nodal variety* of structurally trivial semigroups. A variety $\mathcal{U}$ of structurally trivial semigroups is nodal if there exists some ordered (n,m) of non-negative integers such that for every variety $\mathcal{W}$ formed by groups, we have the equality

$$\mathcal{U} \vee \mathcal{W} = \mathcal{V} \otimes_{(n,m)} \mathcal{W} = \mathcal{V} \oplus_{(n,m)} \mathcal{W}.$$

It is also shown that the set of all nodal varieties is closed under taking varietal meets; and it is still an open problem to determine whether or not the class of all nodal varieties is also closed under taking varietal joins. Examples of such varieties include the following: the variety $\mathcal{N}_n$ of all n-nilpotent semigroups, the variety $\mathcal{T}^{(n,m)}$ of all semigroups S such that $S/\theta(n,m)$ is trivial; and, the variety $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(i,j)}$ of all semigroups S such that $S/\theta(i,j)$ is a n-nilpotent extension of a rectangular band.

If $\mathcal{U}$ is a nodal variety, then we denote by $\mathcal{L}_N(\mathcal{U})$ the lattice of all nodal subvarieties of $\mathcal{U}$. For any variety $\mathcal{W}$ of groups, by $\mathcal{L}_N(\mathcal{U} \vee \mathcal{W})$ we mean the lattice structure of the following partially ordered set:

$$\{\mathcal{X} \vee \mathcal{Y} : \mathcal{X} \in \mathcal{L}_N(\mathcal{U}), \mathcal{Y} \in \mathcal{L}(\mathcal{W})\}.$$

It is proved here that the lattice $\mathcal{L}_N(\mathcal{U} \vee \mathcal{W})$ is isomorphic to the lattice direct product $\mathcal{L}_N(\mathcal{U}) \times \mathcal{L}(\mathcal{W})$. Understanding the lattice structure of $\mathcal{L}_N(\mathcal{U} \vee \mathcal{W})$ is the first step towards understanding the structure of $\mathcal{L}(\mathcal{U} \vee \mathcal{W})$. The lattice $\mathcal{L}(\mathcal{U} \vee \mathcal{W})$ is in general not isomorphic to $\mathcal{L}(\mathcal{U}) \times \mathcal{L}(\mathcal{W})$, and hence our result is a useful estimation of the structure of $\mathcal{L}(\mathcal{U} \vee \mathcal{W})$. The results in this chapter will be generalised further to the varieties formed entirely by dense semilattices of structurally [trivial, group] semigroups in the next chapter.

4:1 STRUCTURALLY GROUP SEMIGROUPS.

A semigroup S is called a *structurally [group, trivial]* semigroup if there exists some ordered pair (n,m) such that the quotient $S/\theta(n,m)$ is a [group, trivial]. In this section we characterise the least structurally trivial congruence τ_s on structurally group semigroups. If $S/\theta(n,m)$ is a group, then it has a unique idempotent $\theta(n,m)$-class which we denoted by:

$$E_{(n,m)} = \{x : (x,x^2) \in \theta(n,m)\}.$$

Lemma 4:1.1 *For any semigroup* S *if* $S/\theta(n,m)$ *is a group, then* S^{n+1+m} *is a rectangular group. Conversely, if* S^n *is a rectangular group, then* $S/\theta(n,n)$ *is a group.*

Proof. Suppose that $S/\theta(n,m)$ is a group, and take any element x of S^{n+1+m}. Then there exists elements $x_1,x_2,\ldots, x_{n+1+m}$ in S such that

$$x = (x_1x_2.\ldots x_n)(x_{n+1})(x_{n+2}x_{n+3}.\ldots x_{n+1+m}).$$

Now, put $a = x_1x_2.\ldots x_n$, $b = x_{n+1}$ and $c = x_{n+2}x_{n+3}.\ldots x_{n+1+m}$, and take any (n,m)-inverses $a' \in V(a;n,m)$, $b' \in V(b;n,m)$, and $c' \in V(c;n,m)$. Then

$$x(c'b'a')x = (abc)(c'b'a')(abc) = ab(cc'b'a'ab)c = abc = x;$$

and so x is regular. The second last equality follows by noting that the $\theta(n,m)$-class of S containing $y = cc'b'a'ab$ is the unique idempotent element of the group $S/\theta(n,m)$. For any idempotent elements e and f of S we have that

$$efe = e^nfe^m = e^nee^m = e;$$

and so S^{n+1+m} is a rectangular band. Conversely, suppose that S^n is rectangular group. Then S is a nilpotent extension of a rectangular group. For convenience, let

$$\rho = \theta(n,n) \cap (S^n \times S^n).$$

As S^n is a rectangular group, it is clear that S^n/ρ is a group since ρ is idempotent pure in the sense that for each idempotent $e \in E(S^n)$, $e\rho = E(S^n)$. Also, since S^n is a rectangular group, we have that for any $x,y \in S^n$ and any idempotent $e \in E(S^n)$, $xey = xy$. From this it follows that for any $a \in S$, we have $xay = xa^n(a^n)^{-1}ay$, where $(a^n)^{-1}$ is the group inverse of the regular element $a^n \in S^n$. Thus for any element a of S, we have $(a,a^n(a^n)^{-1}a) \in \theta(n,m)$. Since the regular element $a^n(a^n)^{-1}a$ belongs to S^n, it follows that $S/\theta(n,n) \cong S^n/\rho$ is a group. ❑

Corollary 4:1.2 *A semigroup S is structurally [group, trivial] if and only if it is a nilpotent extension of a rectangular [group, band].* ❑

Theorem 4.1.3 *Every structurally group semigroup is a subdirect product of a group and a nilpotent extension of a rectangular band.*

Proof. Suppose that $S/\theta(n,m)$ is a group. Then by Lemma 4:1.1 S is a nilpotent extension of the rectangular band S^{n+1+m}. Let

$$\tau_S = \mathcal{H}_S \cap (S^{n+1+m} \times S^{n+1+m}) \cup \{(x,x) : x \in S \setminus S^{n+1+m}\},$$

where $\mathcal{H}_S$ is the Green's relation on S. We will prove that τ_S is a congruence on S. Take any $a,b \in S^{n+1+m}$ such that $(a,b) \in \mathcal{H}_S$. This is equivalent to saying that $(a,b) \in \mathcal{L}_S$ and $(a,b) \in \mathcal{R}_S$. Then for any element s of S, we have $(as,bs) \in \mathcal{L}_S$ since the Green's relation $\mathcal{L}_S$ forms a right congruence. We will show that $(as,bs) \in \mathcal{R}_S$. Now since $(a,b) \in \mathcal{R}_S$ there exist $x,y \in S^{(1)}$ such that $b = ax$ and $a = by$. Then for any s in S we have that $bs = axs$ and $as = bys$. For any $a' \in V(a)$ and any $s' \in V(s;n,m)$ we have that both $a'a$ and ss' belong to $E_{(n,m)}$. Therefore,

$$bs = axs = a(a'a)\,xs = a(a'a)^{n+1+m}xs = a(a'a)^n\,(a'a)(a'a)^m xs$$

$$= a(a'a)^n\,(ss')(a'a)^m xs$$

(since $E_{(n,m)}$ is structurally trivial, ss' and $a'a$ are $\theta(n,m)$-related)

$$= (as)\,(s'a'axs).$$

Also, making use of the second equality $as = bys$, and for any inverse $b' \in V(b)$, we have:

$$as = bys = b(b'\,b)\,xs = b(b'\,b)^{n+1+m}ys = b(b'\,b)^n\,(b'\,b)(b'\,b)^m ys$$

$$= b(b'\,b)^n\,(ss')(b'\,b)^m ys$$

(again, since ss' and $b'\,b$ are $\theta(n,m)$-related)

$$= (bs)\,(s'b'\,bxs).$$

We have thus shown that $(as,bs) \in \mathcal{R}_S$. It follows that $(as,bs) \in \mathcal{H}_S$ and hence τ_S is a right congruence. Dually, one can show that τ_S is also a left congruence. Now, S/τ_S is a nilpotent extension of a rectangular band since τ_S relates all elements within each maximal subgroup, separating elements in different subgroups, and such that all non-regular elements form singleton classes. By the definition of τ_S, to show that $\tau_S \cap \theta(n,m) = \iota_S$, the identity relation on S, it suffices to show that

$$\tau_S \cap (S^{n+1+m} \times S^{n+1+m}) \cap \theta(n,m) = \iota_{S^{n+1+m}}.$$

So let $x,y \in S^{n+1+m}$ be such that $(x,y) \in \theta(n,m)$ and $(x,y) \in \tau_{S^{n+1+m}}$. Since $(x,y) \in \mathcal{H}_S$, the elements x and y belong to the same maximal subgroup and so $x^0 = y^0$. This implies that $x = x^0 x x^0 = y^0 y y^0 = y$. Hence, $\tau_S \cap \theta(n,m) = \iota_S$, and so S is a subdirect product of a group (namely $S/\theta(n,m)$) and a nilpotent extension of a rectangular band (namely S/τ_S). ❑

Lemma 4:1.4 *For any semigroup* S *such that* $S/\theta(n,m)$ *is a group, let*

$$\tau_s = \mathcal{H}_s \cap (S^{n+1+m} \times S^{n+1+m}) \cup \{(x,x) : x \in S \setminus S^{n+1+m}\},$$

where $\mathcal{H}_s$ *is the Green's relation on* S. *Then* τ_S *is the least structurally trivial congruence on* S.

Proof. From Lemma 4:1.1, the ideal S^{n+1+m} is a rectangular group and so the relation τ_s forms a structurally trivial congruence as shown in the proof of Lemma 4:1.3. Now, take any congruence ρ on S such that $R = S/\rho$ is structurally trivial. Then there exists some ordered pair (r,t) such that $\theta^T(t,r) = R \times R$. For any $(a,b) \in \tau_S$, since $(a,b) \in \mathcal{H}_s$ we have $a^0 = b^0$ and so:

$$a\rho = (a^0 a a^0)\rho = (a^0\rho)^r(a\rho)(a^0\rho)^t = (b^0\rho)^r(b\rho)(b^0\rho)^t = b\rho$$

which proves that $(a,b) \in \rho$, showing that $\tau_S \subseteq \rho$. ❑

Corollary 4:1.5 *Let* S *be a semigroup such that* $S/\theta(n,m)$ *is a group. Then for any homomorphism* ϕ *from* S *onto* T, *the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "(\tau_S)^\natural"'] & T \arrow[d, "(\tau_T)^\natural"] \\ S/\tau_s \arrow[r, "\phi_\tau"'] & T/\tau_T \end{tikzcd}$$

where the map ϕ_τ *is defined by* $x\tau_s \mapsto (x\phi)\tau_T$ *for each* $x \in S$.

Proof. Take any homomorphism ϕ from S onto T. Then from Theorem 1:1.4 we see that $T/\theta(n,m)$ is a homomorphic image of $S/\theta(n,m)$. It follows that $T/\theta(n,m)$ is a group and that T^{n+1+m} is a rectangular group. Hence τ_T forms a structurally trivial congruence on T by Lemma 4:1.1 and Lemma 4:1.4. Since $\ker(\phi\circ(\tau_T))$ is a structurally trivial congruence on S, it follows by Lemma 4:1.4 that the diagram commutes. ❑

4:2 AN ALTERNATIVE PROOF

We now give an alternative proof for the structural regularity of all nilpotent extensions of rectangular groups given in Lemma 4:1.1. This proof makes use of the following structure decomposition for extensions of rectangular groups by semigroups with a zero element due to Petrich (1974).

For a non-empty set X, denote by $\mathcal{T}(X)$ (and alternatively, $\mathcal{T}^*(X)$) the semigroup of all mappings from X into itself written as operators on the left (respectively, right). For any $\alpha \in \mathcal{T}(X) \cup \mathcal{T}^*(X)$, if α is a constant map, then by $[\alpha]$ we mean the image of α for any x in X. For a semigroup Q with zero, we denote by Q^* the partial groupoid of all non-zero elements of Q. A partial homomorphism of Q^* into a semigroup T is a mapping φ from Q^* into T such that $(a\varphi)(b\varphi) = (ab)\varphi$ whenever ab is defined in Q^*.

Theorem 4:2.1 (Petrich (1974) Proposition 2.1) *Let L, G, and R be a left zero semigroup, a group, and a right zero semigroup, respectively, let* $P = L \times G \times R$ *and let Q be a semigroup with zero disjoint from P. Let the functions*

$$\begin{array}{ccccc} \mathcal{T}(L) & \xleftarrow{\ \alpha\ } & Q^* & \xrightarrow{\ \beta\ } & \mathcal{T}^*(R) \\ & & \downarrow{\scriptstyle \gamma} & & \\ & & G & & \end{array}$$

given respectively by $q \mapsto \alpha^q$ $(\in \mathcal{T}(L))$, $q \mapsto \beta^q$ $(\in \mathcal{T}^*(R))$, $q \mapsto q^\gamma$ $(\in G)$, *be partial homomorphisms such that both* $\alpha^q\alpha^{q'}$ *and* $\beta^q\beta^{q'}$ *are constant maps if* $qq' = 0$ *in* Q. *Denote the constant values of these maps by* $[\alpha^q\alpha^{q'}]$ *and* $[\beta^q\beta^{q'}]$, *respectively. On* $S = P \cup Q^*$ *define a multiplication* $\bullet$ *as follows:*

(M1) $\qquad (l,g,r)\bullet q = (l\,,\, gq^\gamma, r\beta^q)$

(M2) $\qquad q\bullet(l,g,r) = (\alpha^q l\,,\, q^\gamma g, r)$

(M3) $\qquad q\bullet q' = ([\alpha^q\alpha^{q'}]\,,\, q^\gamma q'^\gamma, [\beta^q\beta^{q'}])$ $\qquad$ *if* $qq' = 0$ *in* Q,

for any $(l\,,\, q\,,\, r)$ *in* P, q,q' *in* Q^*, *and the remaining products are as in* P and Q. *Then S is an ideal extension of* P by Q. *Conversely, every extension of P by Q can be so constructed.* ❑

We point out that if Q is taken to be a nilpotent semigroup in the above construction, then S is a nilpotent extension of a rectangular group.

Corollary 4:2.2 (Corollary 2.2 of Petrich (1974)) *With the notation of Theorem 4:2.1, S is a subdirect product of* T *and* G*, where* $T = (L \times R) \cup Q^*$ *with multiplication* $\bullet$*:*

(M1') $\qquad (l,r)\bullet q = (l\,,\, r\beta^q)$

(M2') $\qquad q\bullet(l,r) = (\alpha^q l\,,\, r)$

(M3') $\qquad q\bullet q' = ([\alpha^q\alpha^{q'}]\,,\,[\beta^q\beta^{q'}])$ $\qquad$ *if* $qq' = 0$ *in* Q,

for any (l,r) *in* $L \times R$, q,q' *in* Q^*, *and the remaining products are as in* $L \times R$ and Q. ❑

Lemma 4:2.3 *If* S^n *is a rectangular group, then* $S/\theta(n,n)$ *is a group.*

Proof. Suppose that S^n is a rectangular group. Then S is a nilpotent extension of a rectangular group. By Theorem 4:2.1 there exists P, Q^*, L and R such that S is constructed as described there, and Q is a n-nilpotent semigroup. Clearly,

$$S^n = L \times G \times R = P.$$

We will show that $S/\theta(n,n)$ is a group. Now, define for each element a of S an element a^* as follows:

$$a^* = \begin{cases} (l,g^{-1},r) & \text{if } a = (l,g,r) \in P \\ (l,(a^\gamma)^{-1},r) & \text{if } a = q \in Q^*, \text{ where } l \text{ and } r \text{ are any fixed} \\ & \qquad \text{elements of } L \text{ and } R \text{ respectively} \end{cases}$$

Note that the elements $(a^\gamma)^{-1}$ and g^{-1} are the inverses of the group elements a^γ and g, respectively. Now, we observe that

$$a\bullet a^*\bullet a = \begin{cases} (l',gg\text{-}1g,r') & , \text{ if } \quad a = (l',g,r') \in P \\ (\alpha^q l, (a^\gamma)(a^\gamma)^{-1}(a^\gamma), r\beta^q) & , \text{ if } a = q \in Q^* \text{ and } a^* = (l\,,(a^\gamma)^{-1},r) \end{cases}$$

$$= \begin{cases} (l',g,r') & , \text{ if } \ a = (l',g,r') \in P \\ (\alpha^q l, a^\gamma, r\beta^q) & , \text{ if } a = q \in Q^* \text{ and } a^* = (l\,,(a^\gamma)^{-1},r) \end{cases}$$

Then for all elements $u = (i,h,j)$ and $v = (t,p,k)$ in $S^n = L \times G \times R = P$,

$$u \bullet a \bullet a^* \bullet a \bullet v = \begin{cases} (i, hgp, k), & \text{if } a = (l',g,r') \in P \\ \\ (i, h(a^\gamma)p, k), & \text{if } a = q \in Q^* \text{ and } a^* = (1, (a^\gamma)^{-1}, r) \end{cases}$$

$$= u \bullet a \bullet v.$$

Similarly, one can show also that $u \bullet a^* \bullet a \bullet a^* \bullet v = u \bullet a^* \bullet v$. We have thus shown that $S/\theta(n,n)$ is regular. The set of all idempotent elements of S is given by $E(S) = \{(x,e,y) : x \in L, y \in R, e^2 = e \in G\}$, and since all the idempotent elements are $\theta(n,n)$-related, $S/\theta(n,n)$ forms a group. ❑

4:3 A PRODUCT OF MAL'TSEV TYPE

Evans (1971) asked if it is possible to define a product on the lattice of all semigroup varieties as is possible for groups and loops? Mal'tsev considered a product for general classes of algebra. We now introduce the first of two products to be investigated in this chapter. For any ordered pair (n,m) of non negative integers, and any varieties of semigroups $\mathcal{U}$ and $\mathcal{W}$, define:

(4:3.1) $\mathcal{U}\oplus_{(n,m)}\mathcal{W} = \{S: S/\theta(n,m) \in \mathcal{W} \text{ and } e\theta(n,m) \in \mathcal{U} \text{ for all } e^2 = e \in S\}$.

In general the above product does not form a variety but for certain special cases it does. This product resembles very much the well known Mal'tsev product defined below:

(4:3.2) $\mathcal{U}\circ\mathcal{W} = \{S: S/\delta \in \mathcal{W}$ and all idempotent δ-classes belong to $\mathcal{U}\}$.

To see the contrast between these products, suppose that $\mathcal{A}$ and $\mathcal{B}$ are respectively, a variety consisting entirely of left groups and the variety of all semilattices. Then $\mathcal{A}\oplus_{(1,0)}\mathcal{B}$ is the variety of all left normal bands while $\mathcal{A}\circ\mathcal{B}$ is a larger class of semigroups containing all semilattices of semigroups in $\mathcal{B}$. This difference occurs because every $\theta(1,0)$-class of a regular semigroup forms a left zero band. In fact, suppose that K is a $\theta(1,0)$-class of a regular semigroup. Then for any $a,b \in K$ and any $a \in V(a)$, $xa = xb$ for all x in S. Since K is an idempotent $\theta(1,0)$-class we have that $a^2 \in K$. Now, we have that

$$a = aa'a = (aa')a^2 = a^2.$$

Also , we see that K is a left zero band since $a = aa = ab$. Thus in general, the above products are distinct and the following equality holds:

$$\mathcal{U}\oplus_{(n,m)}\mathcal{W} = (\mathcal{U} \cap \mathcal{T}^{(n,m)})\oplus_{(n,m)}\mathcal{W}.$$

Lemma 4:3.3 *For any variety $\mathcal{U}$ of structurally trivial semigroups, and any variety $\mathcal{W}$ of semigroups, the class $\mathcal{U}\oplus_{(n,m)}\mathcal{W}$ is closed under taking arbitrary direct products.*

Proof. Let S be the direct product of a family $\{S_\alpha: \alpha\in\Gamma\}$ of semigroups from $\mathcal{U}\otimes_{(n,m)}\mathcal{W}$. We have from Lemma 1:1.9 that $S/\theta^S(n,m)$ is isomorphic to the direct product of the quotients $\{S_\alpha/\theta^{S_\alpha}(n,m) : \alpha\in\Gamma\}$, all of which are assumed to be contained in $\mathcal{W}$. Since $\mathcal{W}$ is a variety, it is closed under arbitrary direct products. Hence, $S/\theta^S(n,m) \in \mathcal{W}$. Also, from Lemma 1:1.9, each idempotent $\theta(n,m)$-class K of S is a direct product of some idempotent $\theta(n,m)$-classes of the semigroups $\{S_\alpha: \alpha\in\Gamma\}$, and each of these idempotent classes is assumed to be in $\mathcal{U}$. Since $\mathcal{U}$ is also closed under direct products, K belongs in $\mathcal{U}$. Thus we have shown that $S \in \mathcal{U}\otimes_{(n,m)}\mathcal{W}$, proving the closure of $\mathcal{U}\otimes_{(n,m)}\mathcal{W}$ under arbitrary direct products. ❑

Lemma 4:3.4 *For any ordered pair (n,m) and any variety $\mathcal{W}$ of semigroups, we have*

$$\mathcal{T}^{(n,m)} \vee \mathcal{W} \subseteq \mathcal{W}^{(n,m)} = \mathcal{T}^{(n,m)}\oplus_{(n,m)}\mathcal{W}.$$

Proof. We have from Theorem 2:2.1(a) that $\mathcal{W}^{(n,m)}$ forms a semigroup variety. As $\mathcal{T}\subseteq\mathcal{W}$ and the map $\mathcal{V} \mapsto \mathcal{V}^{(n,m)}$ preserves class containments, it follows that $\mathcal{T}^{(n,m)} \subseteq \mathcal{W}^{(n,m)}$. It is also clear that $\mathcal{W} \subseteq \mathcal{W}^{(n,m)}$ since $\mathcal{W}$ is closed under homomorphic images. Hence $\mathcal{T}^{(n,m)} \vee \mathcal{W} \subseteq \mathcal{W}^{(n,m)}$.

To prove the equality, take any $S \in \mathcal{W}^{(n,m)}$. By definition $S/\theta(n,m) \in \mathcal{W}$. Take any idempotent $\theta(n,m)$-class K of S. Then for any $a,b \in K$ and for all $u \in K^n$ and $v \in K^m$ we have that $uav = vbv$ and so $K \in \mathcal{T}^{(n,m)}$. This proves that $S \in \mathcal{T}^{(n,m)}\oplus_{(n,m)}\mathcal{W}$, and hence $\mathcal{W}^{(n,m)} \subseteq \mathcal{T}^{(n,m)}\oplus_{(n,m)}\mathcal{W}$. Conversely, for every $S \in \mathcal{T}^{(n,m)}\oplus_{(n,m)}\mathcal{W}$, we have by the definition of the product $\oplus_{(n,m)}$ that $S/\theta(n,m) \in \mathcal{W}$ and so the equality holds. ❑

Lemma 4:3.5 *For each ordered pair (n,m), the lattice map $\mathcal{V}\mapsto \mathcal{V}^{(n,m)}$ given in Theorem 2:2.6 does not map lattice intervals onto lattice intervals in general.*

Proof. The following two inclusions hold:

$$\mathcal{T}^{(1,0)} \subset \mathcal{T}^{(1,0)} \vee \mathcal{S} \subset \mathcal{T}^{(1,0)}\otimes_{(1,0)}\mathcal{S} = \mathcal{S}^{(1,0)}.$$

The second containment holds by Lemma 4:3.4. We have in Example 3:1.8 a semigroup S for which $S/\theta(1,0)$ is a semilattice but S does not belong to the join $\mathcal{T}^{(1,0)} \vee \mathcal{S}$ since for every $k\geq 1$ we have that S^k is never a normal band. Thus we have the strictness of the second containment. The first containment is clearly also strict, and so the interval $[\mathcal{T}^{(1,0)}, \mathcal{S}^{(1,0)}]$ forms a chain of length at least 3, and is therefore not isomorphic to the interval $[\mathcal{T}, \mathcal{S}]$, which is known to be of length two since $\mathcal{S}$ is an atom (see for example, Petrich (1974)). ❑

4:4 VARIETIES OF STRUCTURALLY GROUP SEMIGROUPS

We now introduce the second product. For any variety $\mathcal{U}$ of structurally trivial semigroups, there exists an ordered pair (n,m) of non-negative integers such that $\mathcal{U} \subseteq \mathcal{T}^{(n,m)}$. Then for any variety $\mathcal{W}$ formed entirely by groups, define

$$\mathcal{U} \otimes_{(n,m)} \mathcal{W} = \{S : S/\tau_s \in \mathcal{U}, \text{ and } S/\theta(n,m) \in \mathcal{W}\}$$

where

$$\tau_S = \mathcal{H}_S \cap (S^{n+1+m} \times S^{n+1+m}) \cup \{(x,x) : x \in S \setminus S^{n+1+m}\}.$$

Theorem 4:4.1 *For any structurally trivial variety $\mathcal{U}$ and any variety $\mathcal{W}$ of groups, the class $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ is closed under taking subsemigroups and homomorphic images.*

Proof. To show that $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ is closed under homomorphic images take any semigroup $S \in \mathcal{U} \otimes_{(n,m)} \mathcal{W}$ and any homomorphism ϕ from S onto T. Then from Theorem 1:1.4, the map $\phi_{(n,m)}$ defined by

$$a\theta^S(n,m) \mapsto (a\phi)\theta^T(n,m), \quad (a \in S)$$

is a homomorphism from $S/\theta^S(n,m)$ onto $T/\theta^T(n,m)$; and thus $T/\theta^T(n,m)$ being a homomorphic image of $S/\theta^S(n,m)$, is contained in $\mathcal{W}$. Now we see that T is a structurally group semigroup and that T/τ_T belongs to the variety $\mathcal{U}$ since it is a homomorphic image of S/τ_s by Corollary 4:1.5. This proves that $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ is closed under homomorphic images.

To prove the closure under subsemigroups, take any subsemigroup T of $S \in \mathcal{U} \otimes_{(n,m)} \mathcal{W}$. Denote the $\theta^S(n,m)$-classes of S by $\{S_\alpha : \alpha \in S/\theta(n,m)\}$, and consider the subset of $S/\theta^S(n,m)$ given by

$$\Lambda = \{\alpha : T \cap S_\alpha \neq \varnothing, \alpha \in S/\theta(n,m)\}.$$

Take any $\alpha, \beta \in \Lambda$, and any elements $a \in T \cap S_\alpha$, $b \in T \cap S_\beta$. Since T is a subsemigroup of S, $ab \in T \cap S_{\alpha\beta} \neq \varnothing$; and so $\alpha\beta \in \Lambda$. Therefore Λ belongs to $\mathcal{W}$ since it is a subsemigroup of $S/\theta(n,m)$, which by assumption is contained in $\mathcal{W}$. It is shown in the proof of Theorem 2:1.1(c) that $T/\theta^T(n,m)$ is a homomorphic image of Γ under the map $\alpha \mapsto a\theta^T(n,m)$, if $a \in T \cap S_\alpha \neq \varnothing$; and hence $T/\theta^T(n,m) \in \mathcal{W}$. This proves that T is a structurally group semigroup. To prove that $T \in \mathcal{U}\otimes_{(n,m)}\mathcal{W}$, we need only show that T/τ_T belongs to $\mathcal{U}$. We know that since T^{n+1+m} is a regular subsemigroup of S, and that $T^{n+1+m} \subseteq S^{n+1+m}$,

$$\mathcal{H}_{T^{n+1+m}} = \mathcal{H}_S \cap (T^{n+1+m} \times T^{n+1+m}),$$

from which it follows that:

$$\tau_T = \tau_S \cap (T^{n+1+m} \times T^{n+1+m}) \cup \{(x,x) : x \in T \setminus T^{n+1+m}\}.$$

This means that the map $a\tau_T \mapsto a\tau_S$ for each $a \in T$ is an embedding of T/τ_T into S/τ_S. By assumption S/τ_S belongs to $\mathcal{U}$ and so by the closure of $\mathcal{U}$ under subsemigroups we have $T/\tau_T \in \mathcal{U}$. Thus we have shown that $\mathcal{U}\otimes_{(n,m)}\mathcal{W}$ is closed under subsemigroups. ❑

Counterexample 4:4.2 In general the class $\mathcal{U}\otimes_{(n,m)}\mathcal{W}$ is not closed under taking direct products. Consider for example, the structurally trivial variety

$$\mathcal{U} = [x^2 = y^2, xy = yx, abc = xyz]$$

consisting of certain 3-nilpotent semigroups. The semigroup $A = \{a,b,c,0\}$ with $ab = ba = c$ and all other products equal to 0, belongs to $\mathcal{U}$. Take any variety $\mathcal{W}$ formed entirely of groups containing a non commutative group G. Then it is clear that both A and G belong to $\mathcal{U}\otimes_{(n,m)}\mathcal{W}$. We will show that the direct product $S = A \times G \notin \mathcal{U}\otimes_{(n,m)}\mathcal{W}$. It is easy to see that

$$\tau_S = \{((0,g),(0,h)) : g,h \in G, 0 \in A\} \cup \{((x,y),(x,y)) : 0 \neq x \in A, y \in G\},$$

and that τ_S is precisely the Rees congruence on S with respect to the ideal S^3. It can also be shown that S/τ_S is a 3-nilpotent semigroup. But since G is not commutative by assumption, there exist $s, t \in G$ such that $st \neq ts$; and so we have that

$$(a,s)\,(b,t) = (c,st) \neq (c,ts) = (b,t)\,(a,s)$$

proving that $S/\tau_S \notin \mathcal{U}$. Hence $S = A \times G \notin \mathcal{U}\otimes_{(n,m)}\mathcal{W}$. ❑

We point out that the semigroup S given in Counterexample 4:4.2 above does not belong to $\mathcal{U}\otimes_{(n,m)}\mathcal{W}$ but it does belong to $\mathcal{U}\oplus_{(n,m)}\mathcal{W}$ since the unique idempotent $\theta(n,m)$-class of $S = A \times G$ is just:

$$E_{(n,m)} = \{(x,e) \in S : x \in A, e^2 = e \in G\} \cong A \in \mathcal{U};$$

and so in general these products are distinct. That is, we may have cases where $\mathcal{U}\oplus_{(n,m)}\mathcal{W} \neq \mathcal{U}\otimes_{(n,m)}\mathcal{W}$.

Theorem 4:4.3 *For any variety $\mathcal{W}$ consisting solely of groups and any variety $\mathcal{U}$ of structurally trivial semigroups, there exists some ordered pair (n,m) such that*

$$\mathcal{U} \vee \mathcal{W} \supseteq \mathcal{U} \otimes_{(n,m)} \mathcal{W} \subseteq \mathcal{U} \oplus_{(n,m)} \mathcal{W}.$$

In particular if $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ forms a semigroup variety, then $\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W}$. Further, if $\mathcal{U} \oplus_{(n,m)} \mathcal{W}$ forms a semigroup variety, then the following equalities hold:

$$\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W} = \mathcal{U} \oplus_{(n,m)} \mathcal{W}.$$

Proof. Take any semigroup S in $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$. Then by Theorem 4:1.3, S is a subdirect product of $S/\tau_s \in \mathcal{U}$ and $S/\theta(n,m) \in \mathcal{W}$, proving the first containment. Consider the second containment. For any semigroup S in $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$, we have that $S/\theta(n,m) \in \mathcal{W}$ and that $S/\tau_s \in \mathcal{U}$. To prove that $S \in \mathcal{U} \oplus_{(n,m)} \mathcal{W}$, we need only show that $E_{(n,m)} \in \mathcal{U}$. To do this we will prove that $\tau_s \cap (E_{(n,m)} \times E_{(n,m)})$ is the identity relation on $E_{(n,m)}$. Take any $a,b \in E_{(n,m)}$ such that $(a,b) \in \tau_s$. We observe that $E_{(n,m)}$ is a nilpotent extension of the rectangular band E(S). If a and b are not idempotents then a = b by the definition of τ_s. Otherwise, if a and b are idempotents, then since τ_s separates idempotents, we must have a = b. Hence the map $x \mapsto x\tau_s$ is an embedding of $E_{(n,m)}$ into $S/\tau_s \in \mathcal{U}$, and so $E_{(n,m)}$ is isomorphic to a subsemigroup of $S/\tau_s \in \mathcal{U}$. Now $\mathcal{U}$ being a variety is closed under subsemigroups and so $E_{(n,m)}$ belongs to $\mathcal{U}$, proving the containment $\mathcal{U} \otimes_{(n,m)} \mathcal{W} \subseteq \mathcal{U} \oplus_{(n,m)} \mathcal{W}$.

By substituting $\mathcal{U} = \mathcal{T}$ in the product $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ one can show that $\mathcal{W}$ is contained in $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$. Equally, by substituting $\mathcal{W} = \mathcal{T}$ we have that $\mathcal{U}$ is contained in $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$. If $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ forms a variety, then $\mathcal{U} \vee \mathcal{W} \subseteq \mathcal{U} \otimes_{(n,m)} \mathcal{W}$ and so the equality holds.

Consider now the final set of two equalities. Suppose that $\mathcal{U} \oplus_{(n,m)} \mathcal{W}$ forms a variety of semigroups. To prove the equality $\mathcal{U} \oplus_{(n,m)} \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W}$, take any semigroup S in $\mathcal{U} \oplus_{(n,m)} \mathcal{W}$. Now, the quotient S/τ_s being a homomorphic image of S belongs to $\mathcal{U} \oplus_{(n,m)} \mathcal{W}$ since it is a variety. The quotient S/τ_s is not reductive (unless it is trivial in which case S is a group). This means that $S/\tau_s \notin \mathcal{W}$. So we must have $S/\tau_s \in \mathcal{U}$ since otherwise it would mean that S/τ_s is not structurally trivial. We have thus proved that $\mathcal{U} \oplus_{(n,m)} \mathcal{W} \subseteq \mathcal{U} \otimes_{(n,m)} \mathcal{W}$. The reverse containment holds by what we proved earlier, and so the equality $\mathcal{U} \oplus_{(n,m)} \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W}$ holds. The remaining equality holds since in this case $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ is a variety of semigroups. ❑

We will say a variety $\mathcal{U}$ of structurally trivial semigroups forms a *node* on the lattice of all semigroup varieties, or is a *nodal variety* if there exists some ordered pair (n,m) of non negative integers such that for every variety $\mathcal{W}$ of groups $\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W} = \mathcal{U} \oplus_{(n,m)} \mathcal{W}$. In Examples 4:4.4, 4:4.7 and 4:4.8 we give some concrete examples of such varieties.

Example 4:4.4 For any ordered pair (n,m) of non negative integers and any variety $\mathcal{W}$ of groups, we have:

$$\mathcal{T}^{(n,m)} \vee \mathcal{W} = \mathcal{T}^{(n,m)} \otimes_{(n,m)} \mathcal{W} = \mathcal{T}^{(n,m)} \oplus_{(n,m)} \mathcal{W}.$$

We have from Lemma 4:3.4 that $\mathcal{W}^{(n,m)} = \mathcal{T}^{(n,m)} \oplus_{(n,m)} \mathcal{W}$ forms a variety. By Theorem 4:4.3 this means that $\mathcal{T}^{(n,m)} \otimes_{(n,m)} \mathcal{W} = \mathcal{T}^{(n,m)} \oplus_{(n,m)} \mathcal{W}$ and it follows from the same result that $\mathcal{T}^{(n,m)} \vee \mathcal{W} = \mathcal{T}^{(n,m)} \otimes_{(n,m)} \mathcal{W}$. ❑

Lemma 4:4.5 *For any nodal variety $\mathcal{U}$ and $\mathcal{V}$ of structurally trivial semigroups, the variety $\mathcal{U} \cap \mathcal{V}$ is also a nodal variety.*

Proof. By definition there exist some ordered pairs (n,m) and (r,t) such that for all varieties $\mathcal{W}$ of groups,

$$\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(r,t)} \mathcal{W} = \mathcal{U} \oplus_{(r,t)} \mathcal{W} \text{ and } \mathcal{V} \vee \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W} = \mathcal{U} \oplus_{(n,m)} \mathcal{W}.$$

Let $E_{(n,m)}(S)$ denote the unique idempotent $\theta(n,m)$-class of S if $S/\theta(n,m)$ is a group. Now,

$$\begin{aligned}(\mathcal{U} \vee \mathcal{W}) \cap (\mathcal{V} \vee \mathcal{W}) &= (\mathcal{U} \oplus_{(r,t)} \mathcal{W}) \cap (\mathcal{U} \oplus_{(n,m)} \mathcal{W}) \\ &= \{S: E_{(r,t)}(S) \in \mathcal{U}, S/\theta(r,t) \in \mathcal{W}\} \\ &\cap \{S: E_{(n,m)}(S) \in \mathcal{V}, S/\theta(n,m) \in \mathcal{W}\} \\ &= \{S: E_{(i,j)}(S) \in (\mathcal{U} \cap \mathcal{V}), S/\theta(i,j) \in \mathcal{W}\} \\ &\text{(where } i = \min\{r,n\} \text{ and } j = \min\{t,m\}\text{)} \\ &= (\mathcal{U} \cap \mathcal{V}) \oplus_{(i,j)} \mathcal{W};\end{aligned}$$

and so $(\mathcal{U} \cap \mathcal{V}) \oplus_{(i,j)} \mathcal{W}$ is a variety of semigroups. It follows from Theorem 4:4.3 that

$$(\mathcal{U} \cap \mathcal{V}) \vee \mathcal{W} = (\mathcal{U} \cap \mathcal{V}) \otimes_{(i,j)} \mathcal{W} = (\mathcal{U} \cap \mathcal{V}) \oplus_{(i,j)} \mathcal{W},$$

proving that $\mathcal{U} \cap \mathcal{V}$ is a nodal variety. ❑

Lemma 4:4.6 *For any nodal variety $\mathcal{U}$ of structurally trivial semigroups, the variety*

$$\mathcal{U}^{(n,m)} = \{S: S/\theta(n,m) \in \mathcal{U}\}$$

is also a nodal variety.

Proof. There exists an ordered pair (i,j) such that for any variety $\mathcal{W}$ formed entirely by groups $\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(i,j)} \mathcal{W} = \mathcal{U} \oplus_{(i,j)} \mathcal{W}$. Then for any ordered pair (n,m) we know that $(\mathcal{U} \vee \mathcal{V})^{(n,m)}$ is a variety from Theorem 2:2.6. Now,

$$\begin{aligned}
(\mathcal{U} \vee \mathcal{W})^{(n,m)} &= \{S: S/\theta(n,m) \in (\mathcal{U} \vee \mathcal{W})\} \\
&= \{S: S/\theta(n,m) \in (\mathcal{U} \oplus_{(i,j)} \mathcal{W})\} \\
&= \{S: S/\theta(n,m) \in \{T: T/\theta(i,j) \in \mathcal{W}, \text{ and } E_{(i,j)}(S) \in \mathcal{U}\}\} \\
&= \{S: S/\theta(n+i,m+j) \in \mathcal{W}, \text{ and } E_{(n+i,m+j)}(S) \in \mathcal{U}^{(n,m)}\} \\
&= (\mathcal{U}^{(n,m)}) \oplus_{(n+i,m+j)} \mathcal{W};
\end{aligned}$$

and so $(\mathcal{U}^{(n,m)}) \oplus_{(n+i,m+j)} \mathcal{W}$ is a semigroup variety. It follows from Theorem 4:4.3 that

$$\mathcal{U}^{(n,m)} \vee \mathcal{W} = (\mathcal{U}^{(n,m)}) \otimes_{(n+i,m+j)} \mathcal{W} = (\mathcal{U}^{(n,m)}) \oplus_{(n+i,m+j)} \mathcal{W}$$

and so $\mathcal{U}^{(n,m)}$ is a nodal variety. ❑

Example 4:4.7 Applying Lemma 4:4.5 to the nodal varieties of Example 4:4.4 we can produce more nodal varieties. In fact, every variety of the form $\mathcal{T}^{(n,m)} \cap \mathcal{T}^{(r,t)}$ is a nodal variety. In particular, the variety $\mathcal{N}_k$ of all k-nilpotent semigroups is nodal since $\mathcal{N}_k = \mathcal{T}^{(k-1,0)} \cap \mathcal{T}^{(0,k-1)}$. Proving this equality is equivalent to showing that a semigroup S satisfies the pair of identities

$$z(x_1x_2 \ldots x_{k-1}) = y(x_1x_2 \ldots x_{k-1}) \text{ and } (x_1x_2 \ldots x_{k-1})z = (x_1x_2 \ldots x_{k-1})y$$

if and only if S satisfies the single identity

$$x_1x_2 \ldots x_k = y_1y_2 \ldots y_k.$$

But that is clear from Theorem 2:2.6 since

$$\mathcal{T}^{(k-1,0)} = [(x_1x_2 \ldots x_{k-1})z = (x_1x_2 \ldots x_{k-1})y]$$

and

$$\mathcal{T}^{(0,k-1)} = [z(x_1x_2 \ldots x_{k-1}) = y(x_1x_2 \ldots x_{k-1})].$$

Now, applying Lemma 4:4.6 to $\mathcal{N}_k$, the following variety is also nodal:

$$\begin{aligned}
\mathcal{N}_k^{(i,j)} &= [(x_1x_2 \ldots x_i)(a_1a_2 \ldots a_k)(y_1y_2 \ldots y_j) = (x_1x_2 \ldots x_i)(b_1b_2 \ldots b_k)(y_1y_2 \ldots y_j)] \\
&= \{S: S/\theta(i,j) \in \mathcal{N}_k\}. \quad ❑
\end{aligned}$$

Example 4:4.8 Let $\mathcal{Z}_l = [xy = x]$, $\mathcal{Z}_r = [xy = y]$ and $(\mathcal{Z}_l \vee \mathcal{Z}_r) = [xyx = x]$ denote respectively the varieties of left zero bands, right zero bands, and rectangular bands. Since every [left, right, rectangular] group is a direct product of a [left, right, rectangular] band and a group, the following equalities hold for any variety $\mathcal{W}$ of groups:

$$\mathcal{Z}_l \vee \mathcal{W} = \mathcal{Z}_l \otimes_{(1,0)} \mathcal{W} = \mathcal{Z}_l \oplus_{(1,0)} \mathcal{W}$$

$$\mathcal{Z}_r \vee \mathcal{W} = \mathcal{Z}_r \otimes_{(0,1)} \mathcal{W} = \mathcal{Z}_r \oplus_{(0,1)} \mathcal{W}$$

$$(\mathcal{Z}_l \vee \mathcal{Z}_r) \vee \mathcal{W} = (\mathcal{Z}_l \vee \mathcal{Z}_r) \otimes_{(1,1)} \mathcal{W} = (\mathcal{Z}_l \vee \mathcal{Z}_r) \oplus_{(1,1)} \mathcal{W}.$$

Consequently, the above three varieties of bands are nodal. Applying Lemma 4:4.6 to these varieties, we have that the following are also nodal: $\mathcal{Z}_l^{(i,j)}$, $\mathcal{Z}_r^{(i,j)}$ and $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)\}^{(i,j)}$. ❑

Theorem 4:4.9 *Let $\mathcal{U}$ be a variety of structurally trivial semigroups and $\mathcal{W}$ be a semigroup variety formed entirely of groups. Then for any subvariety $\mathcal{V}$ of the class $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$, we have that*

$$(\mathcal{V} \cap \mathcal{U}) \otimes_{(n,m)} (\mathcal{V} \cap \mathcal{W}) = \mathcal{V} = (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}).$$

Proof. Observe that the class $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ may not be a variety but by assumption $\mathcal{V}$ is. It is clear that

$$\mathcal{V} \supseteq (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}).$$

Take any semigroup S in $\mathcal{V} \subseteq \mathcal{U} \otimes_{(n,m)} \mathcal{W}$. Then S/τ_s belongs in $\mathcal{U}$ and $S/\theta(n,m)$ belongs in $\mathcal{W}$. But $\mathcal{V}$ being a variety is closed under homomorphic images and so $S/\tau_s \in \mathcal{V} \cap \mathcal{U}$ and $S/\theta(n,m) \in \mathcal{V} \cap \mathcal{U}$. From Theorem 4:3.3, S is a subdirect product of S/τ_s and $S/\theta(n,m)$ and so the following two containments hold:

$$\mathcal{V} \subseteq (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W})$$

and

$$\mathcal{V} \subseteq (\mathcal{V} \cap \mathcal{U}) \otimes_{(n,m)} (\mathcal{V} \cap \mathcal{W}).$$

The first two (of the above three) containments prove the equality :

$$\mathcal{V} = (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}) .$$

From Theorem 4:4.3 we have the containment:

$$(\mathcal{V} \cap \mathcal{U}) \otimes_{(n,m)} (\mathcal{V} \cap \mathcal{W}) \subseteq (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}) = \mathcal{V}.$$

This containment, together with the third one above, proves that the equality

$$\mathcal{V} = (\mathcal{V} \cap \mathcal{U}) \otimes_{(n,m)} (\mathcal{V} \cap \mathcal{W});$$

and that completes the proof. ❑

Although not every class of the type $\mathcal{U} \otimes_{(n,m)} \mathcal{W}$ forms a variety, the following result shows that such a decomposition is unique.

Proposition 4:4.10 *For any structurally trivial varieties $\mathcal{U}$ and $\mathcal{V}$ and any varieties $\mathcal{W}$ and $\mathcal{Y}$ such that $\mathcal{U} \otimes_{(r,t)} \mathcal{W} = \mathcal{V} \otimes_{(n,m)} \mathcal{Y}$, then $\mathcal{U} = \mathcal{V}$ and $\mathcal{W} = \mathcal{Y}$.*

Proof. Take any semigroup S in $\mathcal{U} \subseteq \mathcal{U}\otimes_{(r,t)}\mathcal{W} = \mathcal{V}\otimes_{(n,m)}\mathcal{Y}$. Then by the definition of $\otimes_{(n,m)}$ we have that $S/\theta(n,m)$ belongs to $\mathcal{Y}$ and S/τ_s belongs to $\mathcal{V}$. As the semigroup S is in $\mathcal{U}$, it is structurally trivial, and τ_s is the identity relation on S. Therefore $S = S/\tau_s \in \mathcal{V}$, proving that $\mathcal{U} \subseteq \mathcal{V}$. The reverse containment holds similarly and so $\mathcal{U} = \mathcal{V}$.

Next, take any semigroup S in $\mathcal{W} \subseteq \mathcal{U}\otimes_{(r,t)}\mathcal{W} = \mathcal{V}\otimes_{(n,m)}\mathcal{Y}$. Then by the definition of $\otimes_{(n,m)}$ we have that $S/\theta(n,m)$ belongs to $\mathcal{Y}$ and S/τ_s belongs to $\mathcal{V}$. Since $S \in \mathcal{W}$, it is a group and hence reductive. Therefore $\theta(n,m)$ is the identity relation on S. This implies that $S = S/\theta(n,m) \in \mathcal{Y}$, proving that $\mathcal{W} \subseteq \mathcal{Y}$. The reverse containment holds similarly and we have the equality $\mathcal{Y} = \mathcal{W}$. ❑

Lemma 4:4.11 *For any nodal variety $\mathcal{U}$ and any varieties $\mathcal{Y}$ and $\mathcal{W}$ formed entirely by groups, we have:*

$$\mathcal{U} \vee (\mathcal{W} \cap \mathcal{Y}) = (\mathcal{U} \vee \mathcal{W}) \cap (\mathcal{U} \vee \mathcal{Y}).$$

Proof. Since $\mathcal{U}$ is nodal there exists some ordered pair (n,m) such that $\mathcal{U} \oplus_{(n,m)} \mathcal{Y} = \mathcal{U} \vee \mathcal{Y}$ and $\mathcal{U} \oplus_{(n,m)} \mathcal{W} = \mathcal{U} \vee \mathcal{W}$. We will show that $\mathcal{U} \oplus_{(n,m)} (\mathcal{Y} \cap \mathcal{W})$ forms a variety of semigroups:

$$\begin{aligned}(\mathcal{U} \vee \mathcal{Y}) \cap (\mathcal{U} \vee \mathcal{W}) &= (\mathcal{U}\oplus_{(n,m)} \mathcal{Y}) \cap (\mathcal{U} \oplus_{(n,m)} \mathcal{W})\\ &= \{S: S/\theta(n,m) \in \mathcal{Y} \text{ and } E_{(n,m)}(S) \in \mathcal{U}\}\\ &\cap \{S: S/\theta(n,m) \in \mathcal{W} \text{ and } E_{(n,m)}(S) \in \mathcal{U}\}\\ &= \{S: S/\theta(n,m) \in (\mathcal{Y}\cap\mathcal{W}) \text{ and } E_{(n,m)}(S) \in \mathcal{U}\}\\ &= \mathcal{U}\oplus_{(n,m)}(\mathcal{Y} \cap \mathcal{W}).\end{aligned}$$

Since $\mathcal{U}\oplus_{(n,m)}(\mathcal{Y}\cap\mathcal{W})$ is a variety, it follows from Theorem 4:4.3 that

$$\mathcal{U} \vee (\mathcal{Y} \cap \mathcal{W}) = \mathcal{U} \otimes_{(n,m)}(\mathcal{Y} \cap \mathcal{W}) = \mathcal{U}\oplus_{(n,m)}(\mathcal{Y} \cap \mathcal{W});$$

and hence the conclusions follow immediately. ❑

For any nodal variety $\mathcal{V}$ we will denote by $\mathcal{L}_N(\mathcal{V})$ the lattice of all nodal subvarieties of $\mathcal{V}$; and for any variety $\mathcal{W}$ of groups, we will denote by $\mathcal{L}_N(\mathcal{V} \vee \mathcal{W})$ the lattice structure of the following partially ordered set:

$$\{\mathcal{X} \vee \mathcal{Y} : \mathcal{X} \in \mathcal{L}_N(\mathcal{V}), \mathcal{Y} \in \mathcal{L}(\mathcal{W})\}.$$

Theorem 4:4.12 *Fix any nodal variety $\mathcal{U}$ of structurally trivial semigroups, and any variety $\mathcal{W}$ consisting solely of groups. Then the following lattice map is an isomorphism:*

$$\Psi : \mathcal{L}_N(\mathcal{V} \vee \mathcal{W}) \longrightarrow \mathcal{L}_N(\mathcal{V}) \times \mathcal{L}(\mathcal{W}), \qquad \mathcal{V} \mapsto (\mathcal{V} \cap \mathcal{W}, \mathcal{V} \cap \mathcal{W}).$$

The reverse map sends each $(\mathcal{X},\mathcal{Y})$ to $\mathcal{X} \vee \mathcal{Y}$ and hence the lattice $\mathcal{L}_N(\mathcal{V} \vee \mathcal{W})$ is isomorphic to the lattice direct product $\mathcal{L}_N(\mathcal{V}) \times \mathcal{L}(\mathcal{W})$.

Proof. By assumption $\mathcal{U}$ is a nodal variety, and so there exists some ordered pair (n,m) such that $\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W}$. The map Ψ clearly preserves varietal meets. To see that it is one to one, take any subvarieties $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{V} \vee \mathcal{W}$ such that $\mathcal{A}\Psi = \mathcal{B}\Psi$. Then this implies that $\mathcal{A} \cap \mathcal{U} = \mathcal{B} \cap \mathcal{U}$ and $\mathcal{A} \cap \mathcal{W} = \mathcal{B} \cap \mathcal{W}$. From Theorem 4:4.9 we have that

$$\mathcal{A} = (\mathcal{A} \cap \mathcal{U}) \vee (\mathcal{A} \cap \mathcal{W}) = (\mathcal{B} \cap \mathcal{U}) \vee (\mathcal{B} \cap \mathcal{W}) = \mathcal{B}.$$

In view of Lemma 0:4.20, it suffices to show that Ψ is onto.

Take any $(\mathcal{A},\mathcal{B}) \in \mathcal{L}_N(\mathcal{V}) \times \mathcal{L}(\mathcal{W})$. By showing that the following equalities hold, we will in fact be proving the surjectivity of Ψ:

(i) $\quad (\mathcal{A} \vee \mathcal{B}) \cap \mathcal{U} = \mathcal{A}$

(ii) $\quad (\mathcal{A} \vee \mathcal{B}) \cap \mathcal{W} = \mathcal{B}$

Equality (i): Clearly since $\mathcal{A} \subseteq \mathcal{A} \vee \mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{U}$, we have $(\mathcal{A} \vee \mathcal{B}) \cap \mathcal{U} \supseteq \mathcal{A}$. Conversely, take any semigroup S in $(\mathcal{A} \vee \mathcal{B}) \cap \mathcal{U}$. Then since $\mathcal{A} \subseteq \mathcal{U}$ it follows that S is a structurally trivial semigroup. From Theorem 4:4.9 we have that $\mathcal{A} \vee \mathcal{B} = \mathcal{A} \otimes_{(n,m)} \mathcal{B}$ and so $S/\theta(n,m) \in \mathcal{B}$ and $S/\tau_s \in \mathcal{A}$. Since S is structurally trivial S^{n+1+m} is a rectangular band and τ_s is the identity relation on S. This proves that $S = S/\tau_s \in \mathcal{A}$ and so $(\mathcal{A} \vee \mathcal{B}) \cap \mathcal{U} \subseteq \mathcal{A}$ proving the first equality.

Equality (ii): Clearly since $\mathcal{B} \subseteq \mathcal{A} \vee \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{W}$, we have $(\mathcal{A} \vee \mathcal{B}) \cap \mathcal{W} \supseteq \mathcal{B}$. Conversely, take any semigroup S in $(\mathcal{A} \vee \mathcal{B}) \cap \mathcal{W}$. Then since $\mathcal{B} \subseteq \mathcal{W}$ it follows that S is a group. From Theorem 4:4.9 we have that $\mathcal{A} \vee \mathcal{B} = \mathcal{A} \otimes_{(n,m)} \mathcal{B}$ and so $S/\theta(n,m) \in \mathcal{B}$ and $S/\tau_s \in \mathcal{A}$. Since S is a group, it is reductive and so $\theta(n,m)$ is the identity relation on S. This proves that $S = S/\theta(n,m) \in \mathcal{B}$ and so the containment $(\mathcal{A} \vee \mathcal{B}) \cap \mathcal{W} \subseteq \mathcal{B}$ holds. Hence Equality (ii) also holds. ❑

Lemma 4:4.13 *For any nodal variety $\mathcal{U}$ of structurally trivial semigroups, and any variety $\mathcal{W}$ formed entirely of groups, the lattice intervals $[\mathcal{T},\mathcal{W}]$ and $[\mathcal{U},\mathcal{U} \vee \mathcal{W}]$ are isomorphic.*

Proof. By assumption $\mathcal{U}$ is nodal, and so there exists some ordered pair (n,m) such that $\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W}$. We will show that the map $\Phi : [\mathcal{T},\mathcal{W}] \rightarrow [\mathcal{U}, \mathcal{U} \vee \mathcal{W}]$ defined by $\mathcal{X} \mapsto \mathcal{U} \vee \mathcal{X}$, for each $\mathcal{X} \subseteq \mathcal{W}$ is an isomorphism. Take any subvarieties $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{W}$. We see that Φ preserves varietal joins because:

$$\mathcal{U} \vee (\mathcal{A} \vee \mathcal{B}) = (\mathcal{U} \vee \mathcal{A}) \vee (\mathcal{U} \vee \mathcal{B}).$$

It follows from Lemma 4:4.11 that Φ also preserves varietal meets. We will show that Φ is onto. Take any variety $\mathcal{V}$ from the interval $\mathcal{U} \subseteq \mathcal{V} \subseteq (\mathcal{U} \vee \mathcal{W})$. Now, we have from Theorem 4:4.9,

$$\mathcal{V} = (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}) = \mathcal{U} \vee (\mathcal{V} \cap \mathcal{W}) = (\mathcal{V} \cap \mathcal{W})\Phi.$$

We have the second equality since $\mathcal{U} \subseteq \mathcal{V}$ and the third equality follows from the definition of Φ since $\mathcal{V} \cap \mathcal{W} \subseteq \mathcal{W}$. Finally, we will show that it is one-to-one. Take any subvarieties $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{W}$ such that $\mathcal{X}\Phi = \mathcal{Y}\Phi$. This implies that $\mathcal{U} \vee \mathcal{X} = \mathcal{U} \vee \mathcal{Y}$. Then from Theorem 4:4.9 we have

$$\mathcal{U} \otimes_{(n,m)} \mathcal{X} = \mathcal{U} \vee \mathcal{X} = \mathcal{U} \vee \mathcal{Y} = \mathcal{U} \otimes_{(n,m)} \mathcal{Y}.$$

It follows from Proposition 4:4.10 that $\mathcal{X} = \mathcal{Y}$, and so Φ is one-to-one. This completes the proof that Φ is an isomorphism. ❑

Corollary 4:4.14 *The lattice map* $\mathcal{V} \mapsto \mathcal{V}^{(n,m)}$ *of Theorem 2:2.6 is a lattice isomorphism when it is restricted to the subvarieties of a variety formed entirely of groups.* ❑

We close this section with some open problems.

Problem 4:4.15 For any semigroup variety $\mathcal{V}$ consisting entirely of groups, and any nodal subvariety $\mathcal{U}$ of $\mathcal{T}^{(n,m)}$, we have that $\mathcal{U} \otimes_{(n,m)} \mathcal{V} = \mathcal{U} \vee \mathcal{V}$. Given that the identities determining $\mathcal{U}$ and $\mathcal{V}$ are known, it remains an open problem to find a general rule which describes how one can obtain the identities that determine $\mathcal{U} \otimes_{(n,m)} \mathcal{V} = \mathcal{U} \vee \mathcal{V}$. The following particular cases of this have been solved:

(i) if $\mathcal{U}$ is a variety of rectangular bands, then Theorem 2:3.9 describes how one can obtain the identities which determine the variety $\mathcal{U} \otimes_{(n,m)} \mathcal{V}$;

(ii) Theorem 2:2.6 gives the method of obtaining the identities which determine $\mathcal{V}^{(n,m)} = \mathcal{T}^{(n,m)} \otimes_{(n,m)} \mathcal{V} = \mathcal{T}^{(n,m)} \vee \mathcal{V}$; and

(iii) if $\mathcal{U} = \mathcal{N}_n$, the variety of all n-nilpotent semigroups, then in Chapter 6 we show how one can obtain the identities which determine the variety $\mathcal{N}_n \vee \mathcal{V}$ – which consists entirely of all n-inflations of semigroups in $\mathcal{V}$.

The structure theorem below, which in fact is a combination of Theorem 4:1.3 and Theorem 2:2.6, may prove useful in approaching the above problem.

Theorem 4:4.16 *Let* $\mathcal{U} = [u_\alpha = v_\alpha] \subseteq \mathcal{T}^{(n,m)}$ *be a nodal variety and* $\mathcal{W} = [p_\alpha = q_\alpha]$ *be a variety consisting entirely of groups.*

Then

$$\mathcal{U} \vee \mathcal{W} = \{ S\colon S/\theta(n,m) \in \mathcal{W} \textit{ and } S/\tau_S \in \mathcal{U} \}$$

$$= [(p_\alpha , q_\alpha) \in \theta(n,m)] \cap \{S : S/\tau_S \in [u_\alpha = v_\alpha] \}$$

$= \{S$: S *is a subdirect product of* $S/\theta(n,m) \in \mathcal{W}$ *and a semigroup* $T = S/\tau_S$ *in* $\mathcal{U}\}$

$$= \{ S\colon S/\theta(n,m) \in \mathcal{W} \textit{ and } E_{(n,m)}(S) \in \mathcal{U} \} \qquad \square$$

Problem 4:4.17 Let $\mathcal{U}$ be any variety of structurally trivial semigroups, $\mathcal{W}$ be any semigroup variety formed by groups, and $\mathcal{T}^{(\infty,\infty)}$ be the class of all structurally trivial semigroups. Does the class $\mathcal{T}^{(\infty,\infty)} \cap (\mathcal{U} \vee \mathcal{W})$ form a nodal variety of structurally trivial semigroups?

A positive solution to this problem would provide much information about the structure of $\mathcal{L}(\mathcal{U} \vee \mathcal{W})$ in view of Theorem 4:4.12 and Lemma 4:4.13. The answer to the following question would also provide some further information about this lattice.

Problem 4:4.18 Define a relation γ on the set $\mathcal{L}(\mathcal{T}^{(\infty,\infty)})$ of all varieties of structurally trivial semigroups as follows:

$$\gamma = \{(\mathcal{A},\mathcal{B}) : \mathcal{A} \vee \mathcal{W} = \mathcal{B} \vee \mathcal{W} \text{ for all variety } \mathcal{W} \text{ formed by groups}\}.$$

Clearly, γ is an equivalence relation on $\mathcal{L}(\mathcal{T}^{(\infty,\infty)})$. Is it also a congruence? Is it true that each γ-class contains a unique nodal variety, and that the greatest member of each γ-class is that nodal variety?

A positive solution to these question would mean that $\mathcal{L}_N(\mathcal{U}) \cong \mathcal{L}(\mathcal{U})/\gamma$.

CHAPTER 5

VARIETIES OF STRUCTURALLY INVERSE SEMIGROUPS

In Chapter 4 we introduced two new products on the lattice of all semigroup varieties. It is shown here using one of the products, namely the product $\oplus_{(n,m)}$ which resembles the Mal'tsev-product, that every semigroup variety $\mathcal{V}$ formed entirely by *dense semilattices of structurally [trivial, group] semigroups* can be uniquely expressed as a product $\mathcal{V} = (\mathcal{U}\oplus_{(n,m)} \mathcal{G}) \vee \mathcal{X}$, where $\mathcal{U}$ is a semigroup variety of structurally trivial semigroups, $\mathcal{X}$ is a variety of semilattices, and $\mathcal{G}$ is a semigroup variety formed by groups.

Recall that a semigroup S is *structurally inverse* if the quotient $S/\theta(n,m)$ is an inverse semigroup for some (n,m); and a semigroup S is an *n-nilpotent extension* of a semigroup T if $S^n = T$, for some $n \geq 1$. It was shown in Lemma 0:4.17 that for any variety $\mathcal{V}$, the class $\mathcal{V}^n$ of all n-nilpotent extensions of semigroups in $\mathcal{V}$ also forms a variety.

We will, as before, let $(\mathcal{Z}_l \vee \mathcal{Z}_r) = [xyx = x]$ denote the variety of all *rectangular bands*, and $\mathcal{S} = [x^2 = x, xy = yx]$ be the variety of all *semilattices*. Petrich (1974) proved that for any variety $\mathcal{W} \subseteq [x^{n+1} = x]$, consisting entirely of groups, the lattice $\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^2 \vee \mathcal{S} \vee \mathcal{W})$ of all subvarieties of the join

$$(\mathcal{Z}_l \vee \mathcal{Z}_r)^2 \vee \mathcal{S} \vee \mathcal{W},$$

where $(\mathcal{Z}_l \vee \mathcal{Z}_r)^2$ denotes the variety of all 2-nilpotent extensions of rectangular bands, is isomorphic to the direct product

$$\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^2) \times \mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{W}).$$

Unfortunately, we are not able to generalise Petrich's results exactly, but are able to prove something that resembles his results. In fact, we are only able to generalise the results of Chapter 4.

Recall that a semigroup S is called a *structurally trivial* if $S/\theta(n,m)$ is trivial for some (n,m); and equivalently such semigroups are nilpotent extensions of rectangular bands. A sketch of the lattice of all semigroup varieties consisting solely of structurally trivial semigroups is given in Chapter 7. *Generalised inverse semigroups* — those regular semigroups whose idempotent elements satisfy the identity efgh = egfh — are precisely the regular semigroups which are structurally inverse.

We defined a variety consisting entirely of structurally trivial semigroups to be a *nodal* if there exists some (n,m) such that for any variety $\mathcal{W}$ of groups $\mathcal{U}\otimes_{(n,m)}\mathcal{W} = \mathcal{U}\oplus_{(n,m)}\mathcal{W} = \mathcal{U} \vee \mathcal{W}$; and we denoted by $\mathcal{L}_N(\mathcal{V})$ the lattice of all nodal subvarieties of $\mathcal{V}$. In this chapter we will consider varieties of the form

$$(\mathcal{U}\oplus_{(n,m)}\mathcal{W}) \vee \mathcal{S} = (\mathcal{U} \vee \mathcal{W}) \vee \mathcal{S},$$

where $\mathcal{S}$ is the variety of all semilattices. The variety $\mathcal{U} \vee \mathcal{W}$ consists entirely of strong semilattices of groups, and for any nodal variety $\mathcal{U}$, by $\mathcal{L}_N(\mathcal{U} \vee \mathcal{Z})$ we mean the lattice structure of the partially ordered set:

$$\{\mathcal{A} \vee \mathcal{B} : \mathcal{A} \in \mathcal{L}_N(\mathcal{V}),\ \mathcal{B} \in \mathcal{L}(\mathcal{Z})\}.$$

We generalise both Petrich's results and our results in Chapter 4 by showing that for every $n \geq 1$:

$$\mathcal{L}_N((\mathcal{Z}_1 \vee \mathcal{Z}_r)^n \vee \mathcal{S} \vee \mathcal{W}) \cong \mathcal{L}_N((\mathcal{Z}_1 \vee \mathcal{Z}_r)^n) \times \mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{W}).$$

5:1 NILPOTENT EXTENSIONS OF ORTHODOX NORMAL BANDS OF GROUPS

A semigroup is said to be an *orthodox normal band of groups* if it is a generalised inverse semigroup for which there exist a congruence Θ such that every Θ-class is a group. In this section, we will characterise semigroup varieties of structurally inverse semigroups which are also nilpotent extensions of regular semigroups. It turns out that such semigroup varieties consist entirely of either of the following types: *dense semilattices of structurally [trivial, group] semigroups.*

For any class $\mathcal{V}$ of semigroups, by $\mathcal{V}^{(\infty)}$ we denote the set of all nilpotent extensions of members of $\mathcal{V}$, namely $\{S: S^n \in \mathcal{V}$ for some $n \geq 1\}$. Throughout this chapter $\mathcal{R}$ shall denote the class of all regular semigroups, and by $\mathcal{R}^{(\infty)}$ we will mean the class of all nilpotent extensions of regular semigroups. We have from Examples 3:1.8 and 3:1.14 that the class of all structurally regular (inverse) semigroups is not properly contained in the class of all nilpotent extensions of regular semigroups. The following counter example, constructed by N. Ruskuc (private communication), shows that not every nilpotent extension of a regular (inverse) semigroup is structurally regular. Hence, the class $\mathcal{R}^{(\infty,\infty)}$ of all structurally regular semigroups, and the class $\mathcal{R}^{(\infty)}$ of all nilpotent extensions of regular semigroups are not comparable, in the sense that neither class contains the other.

Example 5:1.1 (Nikola Ruskuc) Denote by R a subsemigroup of the semigroup of all full transformations on a 5-element set, generated by the following three mappings:

$$a = \begin{pmatrix} 12345 \\ 25455 \end{pmatrix} \qquad x = \begin{pmatrix} 12345 \\ 51555 \end{pmatrix} \qquad y = \begin{pmatrix} 12345 \\ 55535 \end{pmatrix}.$$

One can show that R has exactly 10 elements. The following element

$$a^2 = \begin{pmatrix} 12345 \\ 55555 \end{pmatrix} = z$$

turns out to be the zero of R since: $z = az = za = xz = zx = zy = yz$ The remaining 6 elements of R are the following mappings:

$$ax = \begin{pmatrix} 12345 \\ 15555 \end{pmatrix}, \qquad ay = \begin{pmatrix} 12345 \\ 55355 \end{pmatrix}, \qquad xa = \begin{pmatrix} 12345 \\ 52555 \end{pmatrix},$$

$$ya = \begin{pmatrix} 12345 \\ 55545 \end{pmatrix}, \qquad axa = \begin{pmatrix} 12345 \\ 25555 \end{pmatrix}, \qquad aya = \begin{pmatrix} 12345 \\ 55455 \end{pmatrix}.$$

Table 5:1.2 The Cayley table of R

	a	x	y	xa	ax	ya	ay	axa	aya	z
a	z	ax	ay	axa	z	aya	z	z	z	z
x	xa	z	z	z	x	z	z	xa	z	z
y	ya	z	z	z	z	z	y	z	ya	z
xa	z	x	z	xa	z	z	z	z	z	z
ax	axa	z	z	z	ax	z	z	axa	z	z
ya	z	z	y	z	z	ya	z	z	z	z
ay	aya	z	z	z	z	z	ay	z	aya	z
axa	z	ax	z	axa	z	z	z	z	z	z
aya	z	z	ay	z	z	aya	z	z	z	z
z	z	z	z	z	z	z	z	z	z	z

We observe that R is not a regular semigroup, since the element a is not regular. Moreover, R is a 2-nilpotent extension of the regular subsemigroup $R^2 = R\backslash\{a\}$. The following verification shows that the subsemigroup R^2 is regular:

$$xax = x, \quad yay = y, \quad (xa)^2 = xa, \quad (ax)^2 = ax, \quad (ya)^2 = ya, \quad (ay)^2 = ay,$$

$$(axa)x(axa) = axa, \quad (aya)y(aya) = aya \quad \text{and} \quad z^2 = z.$$

Moreover, since the idempotent elements $E(R^2) = \{xa, ya, ax, ay, z\} = E(R)$ commute, R^2 forms an inverse semigroup, and so R is a nilpotent extension of the inverse subsemigroup R^2. Since no two rows or two columns on the table above are identical, both $\theta(1,0)$ and $\theta(0,1)$ reduce to the identity relation on R. This means that R is a reductive semigroup. Hence it follows that $R = R/\theta(n,m)$ for every n and m, proving that R is not structurally regular. Thus we have shown that a nilpotent extension of a regular (inverse) semigroup is not necessarily structurally regular. ❑

The following result of Melnik (1970) will play a crucial role in characterising semigroup varieties which consist entirely of structurally inverse semigroups and which at the same time are nilpotent extensions of regular semigroups. This same result was used in Petrich (1974) to prove the very same results that we generalise in this chapter.

Theorem 5:1.3 (Melnik (1970)) *Let r be a positive integer and let $\mathcal{V}$ be a subvariety of the equational class* $[x^r = (x^r y^r x^r)^r]$. *Let* $S = \{\bigcup S_\alpha : \alpha \in \Gamma\}$ *be a semigroup which is a semilattice Γ of semigroups S_α, each of which is contained in $\mathcal{V}$. Assume that for any $\alpha, \beta \in \Gamma$ such that $\alpha \geq \beta$ there exists a function $\phi_{\alpha,\beta}: S_\alpha \to S_\beta$ and that these functions satisfy: for* $a \in S_\alpha$, $b \in S_\beta$

(C1) $a\phi_{\alpha,\alpha} = a^r$

(C2) $a^2\phi_{\alpha,\beta} = a\phi_{\alpha,\beta}$

(C3) $(a\phi_{\alpha,\beta})\phi_{\beta,\gamma} = a\phi_{\alpha,\gamma}$

(C4) $(a\phi_{\alpha,\beta})b = a^r b$

(C5) $b(a\phi_{\alpha,\beta}) = ba^r$

Then $S \in \mathcal{V} \vee \mathcal{S}$. Conversely, every semigroup in $\mathcal{V} \vee \mathcal{S}$ can be so constructed. ❑

We will follow the convention of Melnik (as pointed out in Petrich (1974)), and refer to the semigroup S constructed in the above result as a *dense semilattice* of the semigroups S_α .

For any class $\mathcal{C}$ of semigroups, we will denote the class of all homomorphic images, subsemigroups, and arbitrary direct products of members of $\mathcal{C}$, respectively, by $H(\mathcal{C})$, $S(\mathcal{C})$ and $P(\mathcal{C})$.

Theorem 5:1.4 *For any semigroup variety $\mathcal{W}$ consisting solely of groups, any nodal subvariety $\mathcal{U}$ of $\mathcal{T}^{(n,m)}$, and the variety $\mathcal{S}$ of all semilattices,*

$$\mathcal{U} \vee \mathcal{S} \vee \mathcal{W} = (\mathcal{U} \oplus_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}.$$

Proof. By assumption $\mathcal{U} \subseteq \mathcal{T}^{(n,m)}$ is nodal and so there exist some ordered pair (n,m) such that $\mathcal{U}\oplus_{(n,m)}\mathcal{W} = \mathcal{U} \vee \mathcal{W}$. By the associativity of the operation of taking varietal joins $\vee$, we have

$$(\mathcal{U}\oplus_{(n,m)}\mathcal{W}) \vee \mathcal{S} = (\mathcal{U} \vee \mathcal{W}) \vee \mathcal{S} = \mathcal{U} \vee \mathcal{W} \vee \mathcal{S}.$$

We will first show that

$$(\mathcal{U}\oplus_{(n,m)}\mathcal{W}) \vee \mathcal{S} \subseteq \mathcal{U}\oplus_{(n,m)}(\mathcal{W} \vee \mathcal{S}).$$

For any semigroup S in $\mathcal{U} \vee \mathcal{W} \vee \mathcal{S}$, let K be a non trivial idempotent $\theta(n,m)$-class of S. Now, K belongs to $\mathcal{U} \vee \mathcal{W} \vee \mathcal{S}$ by its closure under taking subsemigroups. As K is a non-trivial class, there exist elements $a,b \in K$ such that $a \neq b$. This implies that $\theta^K(n,m)$ is not the identity relation on K. If K were reductive, then by definition both $\theta^K(1,0)$ and $\theta^K(0,1)$ would be equal to the identity relation on K. This implies that every $\theta^K(i,j)$ is the identity relation on K by Lemma 1:1.2. But since K is non-trivial, $\theta^K(n,m)$ is not the identity relation, and so K is not reductive. Therefore, $K \notin \mathcal{W} \vee \mathcal{S}$ since $\mathcal{W} \vee \mathcal{S}$ consist of inverse semigroups. This means that either $K \in \mathcal{U}$ or $K/\theta(n,m) \in \mathcal{W} \vee \mathcal{S}$. The latter case would imply that K is not structurally trivial and so we must have $K \in \mathcal{U}$.

Next we will show that $S/\theta(n,m)$ belongs to $\mathcal{W} \vee \mathcal{S}$ for every member S of $\mathcal{U} \vee \mathcal{W} \vee \mathcal{S}$. Now, since $\mathcal{U} \vee \mathcal{W} \vee \mathcal{S} = (\mathcal{U}\oplus_{(n,m)}\mathcal{W}) \vee \mathcal{S}$ forms a variety of semigroups, we have (by Mal'tsev (1937)) that

$$\mathrm{HSP}((\mathcal{U}\oplus_{(n,m)}\mathcal{W}) \vee \mathcal{S}) = (\mathcal{U}\oplus_{(n,m)}\mathcal{W}) \vee \mathcal{S},$$

and so every member of the join $(\mathcal{U}\oplus_{(n,m)}\mathcal{W}) \vee \mathcal{S}$ is a homomorphic image of some subsemigroup of a direct product of some members of $\mathcal{U}\oplus_{(n,m)}\mathcal{W}$ and $\mathcal{S}$. Without loss of generality, take any semigroup $A \in \mathcal{U}\oplus_{(n,m)}\mathcal{W}$, $B \in \mathcal{S}$ and consider the direct product $S = A \times B$. Then by Lemma 1:1.9,

$$S/\theta(n,m) \cong A/\theta(n,m) \times B/\theta(n,m) \cong A/\theta(n,m) \times B \in \mathcal{W} \vee \mathcal{S}$$

since B is reductive and $A/\theta(n,m) \in \mathcal{W}$. We have thus shown that $S \in \mathcal{U}\oplus_{(n,m)}(\mathcal{W} \vee \mathcal{S})$. Take any subsemigroup T of S. As shown in the proof of Lemma 2:1.1(c), there exists a subsemigroup of $S/\theta(n,m)$ (denoted there by Λ) such that $T/\theta(n,m)$ is a homomorphic image of Λ. It follows by the closure of $\mathcal{W} \vee \mathcal{S}$ under homomorphic images and subsemigroups, that $T/\theta(n,m) \in \mathcal{W} \vee \mathcal{S}$. Now, take any homomorphic image R of T. Since $R/\theta(n,m)$ is a homomorphic image of $T/\theta(n,m)$, it follows by Theorem 1:1.4 that $R/\theta(n,m) \in \mathcal{W} \vee \mathcal{S}$. We have thus proved the containment:

$$(\mathcal{U}\oplus_{(n,m)}\mathcal{W}) \vee \mathcal{S} \subseteq \mathcal{U}\oplus_{(n,m)}(\mathcal{W}\vee\mathcal{S}).$$

Next we will prove that

$$\mathcal{U}\vee \mathcal{S}\vee \mathcal{W}\subseteq \mathcal{R}^{(\infty)}.$$

Equivalently, we will show that every member of $\mathcal{U}\vee \mathcal{S}\vee \mathcal{W}$ is a nilpotent extension of some regular semigroup. These, together, will imply that

$$\mathcal{U}\vee\mathcal{S}\vee \mathcal{W}\subseteq (\mathcal{U}\oplus_{(n,m)} (\mathcal{S}\vee\mathcal{W}))\bigcap \mathcal{R}^{(\infty)}.$$

Again, since $\mathcal{U}\vee\mathcal{S}\vee \mathcal{W}$ forms a semigroup variety, we have that

$$\mathrm{HSP}(\mathcal{U}\vee \mathcal{S}\vee \mathcal{W}) = \mathcal{U}\vee \mathcal{S}\vee \mathcal{W},$$

and so every member of the join $\mathcal{U}\vee\mathcal{S}\vee\mathcal{W}$ is a homomorphic image of some subsemigroup of a direct product of some members in $\mathcal{U}$, $\mathcal{S}$, and $\mathcal{W}$. Now, without loss of generality, consider a direct product $S = A \times B \times C$, where $A \in \mathcal{U}$, $B \in \mathcal{S}$ and $C \in \mathcal{W}$. Since $A \in \mathcal{U}$, and $\mathcal{U}\subseteq \mathcal{T}^{(n,m)}$ consists entirely of nilpotent extensions of rectangular bands by Lemma 4:1.1, $A^{n+1+m} = D$ is a rectangular band.

Then

$$S^{n+1+m} = (A\times B\times C)^{n+1+m} = A^{n+1+m}\times B\times C = D\times B\times C$$

is a regular semigroup, since the class of all regular semigroups is closed under taking direct products. Hence S is a (n+1+m)-nilpotent extension of the regular semigroup S^{n+1+m}. Now, take any subsemigroup T of S and define:

$$A_1 = \{x: (x,y,z)\in T\},\quad B_1 = \{y: (x,y,z)\in T\}\quad \text{and}\quad C_1 = \{z: (x,y,z)\in T\}.$$

One can show also that A_1, B_1 and C_1 are subsemigroups of A, B and C, respectively. In particular, the subsemigroup A_1^{n+1+m} of A_1 is a rectangular band, while B_1 and C_1 are regular semigroups.

For each $i \geq 1$, define

$$T_A{}^i = \{(x,y,z) : x \in A_1{}^i, y\in B, z \in C\}$$

$$T_B{}^i = \{(x,y,z) : x \in A, y \in B_1{}^i, z \in C\}$$

$$T_C{}^i = \{(x,y,z) : x \in A, y \in B, z \in C_1{}^i\}.$$

Then

$$T^i = T_A{}^i \bigcap T_B{}^i \bigcap T_C{}^i .$$

For each $a = (x,y,z) \in T^{n+1+m}$, the element $a' = (x',y',z')$ is an inverse of a, where x', y', z' are inverses of x,y,z, respectively, in the regular semigroups $A_1{}^{n+1+m}$, B_1 and C_1. Thus we have shown that T is a (n+1+m)-nilpotent extension of the regular subsemigroup T^{n+1+m}.

For any homomorphism ϕ from T onto N,

$$N^{n+1+m} = (T\phi)^{n+1+m} = (T^{n+1+m})\phi.$$

Thus we have shown that N is a (n+1+m)-nilpotent extension of the regular subsemigroup N^{n+1+m}. Hence every member of the join $\mathcal{U} \vee \mathcal{S} \vee \mathcal{W}$ is a nilpotent extension of some regular semigroup and hence $\mathcal{U} \vee \mathcal{S} \vee \mathcal{W} \subseteq \mathcal{R}^{(\infty)}$. From this fact, together with the containment we proved earlier, it follows that the following inclusion also holds:

$$\mathcal{U} \vee \mathcal{S} \vee \mathcal{W} \subseteq (\mathcal{U} \oplus_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}.$$

We will prove the reverse containment by making use of Theorem 5:1.3, after showing that every semigroup in $(\mathcal{U} \oplus_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}$ can be partitioned into a family of pairwise disjoint semigroups from $\mathcal{U} \vee \mathcal{W} = \mathcal{U} \oplus_{(n,m)} \mathcal{W}$, indexed by a semilattice, and such that all the conditions listed in Theorem 5:1.3 are satisfied. That is, we will show that every semigroup in $(\mathcal{U} \oplus_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}$ is a dense semilattice of semigroups in $\mathcal{U} \vee \mathcal{W}$.

Accordingly, take any semigroup S from $(\mathcal{U} \oplus_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}$. Then, by the definition of the product $\oplus_{(n,m)}$, the quotient $S/\theta(n,m) \in (\mathcal{S} \vee \mathcal{W})$ and $e\theta(n,m) \in \mathcal{U}$ for all $e^2 = e \in S$. As pointed out in Theorem XII.4.3 of Petrich (1984), every semigroup in $\mathcal{S} \vee \mathcal{W}$ is a (strong) semilattice of groups from $\mathcal{W}$. It follows, therefore, that since $S/\theta(n,m) \in \mathcal{S} \vee \mathcal{W}$, there exists a semilattice Γ and a family of pairwise disjoint groups $\{G_\alpha: \alpha \in \Gamma\}$ in $\mathcal{W}$ such that

$$S/\theta(n,m) = \bigcup \{G_\alpha: \alpha \in \Gamma\}.$$

Now, for each $\alpha \in \Gamma$ define

$$S_\alpha = \{x \in S : x\theta(n,m) \in G_\alpha\},$$

the union of all $\theta(n,m)$-classes of S indexed by elements of the group G_α. Then $S = \bigcup \{S_\alpha: \alpha \in \Gamma\}$. To see that each S_α forms a subsemigroup of S, take any $x, y \in S_\alpha$ and suppose that $x\theta(n,m) = \gamma$ and $y\theta(n,m) = \delta$. Then since both $\gamma, \delta \in G_\alpha \subseteq S/\theta(n,m)$, it follows that $(xy)\theta(n,m) = \gamma\delta \in G_\alpha$ and so $xy \in S_\alpha$.

Since $S/\theta(n,m)$ is regular, the set Reg(S) of all regular elements forms a (regular) subsemigroup of S by Lemma 3:1.5. For any $e,f,g,h \in E(S)$,

$$efgh = e^n fgh^m = e^n gfh^m = egfh,$$

since the idempotent $\theta(n,m)$-classes commute in S. We have thus shown that Reg(S) is a generalised inverse semigroup. Moreover, we see that each $E(S_\alpha)$ is a rectangular band, since for any $e,f \in E(S_\alpha)$,

$$efe = e^n fe^m = e^n ee^m = e,$$

because the idempotent elements in S_α are all $\theta(n,m)$-related. Hence each $\mathrm{Reg}(S_\alpha)$ is a rectangular group. One can show that the quotient $S_\alpha/\theta^{S_\alpha}(n,m)$ is isomorphic to the group G_α. Since $\mathcal{S} \vee \mathcal{W}$ consists entirely of completely regular semigroups, we have by Proposition 0:4.22 that there exists some $k \geq 1$ such that $\mathcal{S} \vee \mathcal{W} \subseteq [x^{k+1} = x]$.

Each S_α is a structurally group semigroup since $S_\alpha/\theta(n,m) \cong G_\alpha$ and the unique idempotent $\theta(n,m)$-class lies in $\mathcal{U}$. Hence S_α belongs to $\mathcal{U} \oplus_{(n,m)} \mathcal{W}$. In the following result, we show that $\mathcal{U} \oplus_{(n,m)} \mathcal{W} \subseteq [(x^r y^r x^r)^r = x^r]$. This inclusion will be necessary later when we come to using Theorem 5:1.3.

Before proceeding with the proof, we pause to prove the following useful result.

Lemma 5:1.5 *With the notations of Theorem 5:1.4, let* $r = (n+1+m)k$. *Then the following containment holds:*

$$\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W} = \mathcal{U} \oplus_{(n,m)} \mathcal{W} \subseteq [(x^r y^r x^r)^r = x^r].$$

Proof. By assumption $\mathcal{U} \subseteq \mathcal{T}^{(n,m)}$ is a nodal variety and $\mathcal{W} \subseteq [x^{k+1} = x]$ is a semigroup variety of groups, and so we have:

$$\mathcal{U} \vee \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W} = \mathcal{U} \oplus_{(n,m)} \mathcal{W}.$$

Now, for any semigroup T in $\mathcal{U} \vee \mathcal{W}$, the quotient $T/\theta^T(n,m)$ belongs to $\mathcal{W} \subseteq [x^{k+1} = x]$. From Lemma 3:1.4, this implies that T satisfies the identity

$$x^{n+1+m} = x^{(n+1+m)(k+1)}.$$

We see that for each element a of T, and $r = (n+1+m)k$, a^r is an idempotent element since:

$$(a^r)^2 = a^r\ a^r = a^{(n+1+m)k}\ a^{(n+1+m)k} = a^{(n+1+m)k + (n+1+m)}\ a^{(n+1+m)(k-1)}$$

$$= a^{(n+1+m)(k+1)}\ a^{(n+1+m)(k-1)} = a^{(n+1+m)}\ a^{(n+1+m)(k-1)}$$

$= a^{(n+1+m)k} = a^r.$

Since semigroups in $\mathcal{U} \vee \mathcal{W}$ are nilpotent extensions of rectangular groups (from Lemma 4:1.1), the set $E(T) = \{a^r : a \in T\}$ forms a rectangular band, and so T satisfies the identity $x^r y^r x^r = x^r$. It follows that since E(T) consists of idempotent elements, T also satisfies $(x^r y^r x^r)^r = x^r$. ❑

We now return to the proof of Theorem 5:1.4. For each $\alpha \geq \beta$ define a map between the structurally group subsemigroups S_α and S_β as follows:

$$\phi_{\alpha,\beta}: S_\alpha \to S_\beta,\ x \mapsto x^r b x^r, \quad \text{for any idempotent element b of } S_\beta.$$

We see that this map is well defined since the idempotent elements of S_β are all $\theta(n,m)$-related in S. We will check that the conditions C1 - C5 of Theorem 5:1.3 are all satisfied by this system of subsemigroups $\{S_\alpha : \alpha \in \Gamma\}$ and the family of maps $\{\phi_{\alpha,\beta} : \alpha \geq \beta,\ \alpha, \beta \in \Gamma\}$.

(C1): For any element a in S_α $\quad a\phi_{\alpha,\alpha} = a^r(a^r)a^r = a^r$;

(C2): For any $\alpha \geq \beta$, and any $a \in S_\alpha$, $f \in E(S_\beta)$

$$a^2\phi_{\alpha,\beta} = (a^2)^r f (a^2)^r = a^{2r} f a^{2r} = a^r f a^r = a\phi_{\alpha,\beta};$$

and so (C2) holds.

(C3): For any $\alpha \geq \beta \geq \gamma$, $a \in S_\alpha$, $f \in E(S_\beta)$ and $g \in E(S_\gamma)$

$$(a\phi_{\alpha,\beta})\phi_{\beta,\gamma} = (a^r f a^r)\phi_{\beta,\gamma} = (a^r f a^r)^r g (a^r f a^r)^r = a^r [(a^r f a^r)^r g (a^r f a^r)^r] a^r = a\phi_{\alpha,\gamma}.$$

The last equality follows from the observation that $x = (a^r f a^r)^r g (a^r f a^r)^r$ is an idempotent element, as it is a product of the idempotent elements $(a^r f a^r)^r$ and g in the generalised inverse semigroup Reg(S). That x is contained in S_γ follows from the fact that $\alpha \geq \beta \geq \gamma$. This proves the last equality, and we have the transitivity condition (C3).

Before proceeding with the proof, again we pause to point out the following lemma, another useful property of the regular elements in S.

Lemma 5:1.6 *With the notation of* Theorem 5:1.4 *and* Lemma 5:1.5, *for every semigroup* S *in the class* $(\mathcal{U} \oplus_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}$, *the subsemigroup* Reg(S) *is contained in the equational class* $[x^{r+1} = x]$.

Proof. We have shown that Reg(S) forms a generalised inverse subsemigroup. We will prove here that it satisfies the identity $x^{r+1} = x$, where $r = (n+1+m)k$. As assumed, the quotient $S/\theta(n,m)$ satisfies the identity $x^{k+1} = x$; and so it follows that for every $a \in S$, $(a,a^{k+1}) \in \theta(n,m)$. Thus for any $a \in \mathrm{Reg}(S)$ and any $a' \in V(a)$,

$$a = (aa')^n a(a'a)^m = (aa')^n a^{k+1}(a'a)^m = a^{k+1}.$$

This implies that a^k is an idempotent element of S. Now,

$$a^{r+1} = a^{(n+1+m)k+1} = a^{(n+1+m)k}\, a = (a^k)^{(n+1+m)}\, a = a^{k+1} = a. \qquad \square$$

We now return to the proof of Theorem 5:1.4. Recall that by assumption, S belongs to the class-intersection $(\mathcal{U}\oplus_{(n,m)}(\mathcal{S}\vee\mathcal{W})) \cap \mathcal{R}^{(\infty)}$, where $\mathcal{R}^{(\infty)}$ denotes the class of all nilpotent extensions of regular semigroups. It follows therefore, that the generalised inverse subsemigroup Reg(S) is an ideal of S since $\mathrm{Reg}(S) = S^t$, for some $t \geq 1$ by assumption.

C4. For any $\alpha \geq \beta$, and any $a \in S_\alpha$, $b \in S_\beta$

$$\begin{aligned}
a^r b &= (a^r b)^{r+1} && \text{(by Lemma 5:1.6 since } a^r b \text{ is regular)}\\
&= (a^r b)\,(a^r b)^{r-1}\,(a^r b)\\
&= (a^r a^r b)\,(a^r b)^{r-1}\,(a^r b) && \text{(since } a^r a^r = a^r \text{ from Lemma 5:1.5)}\\
&= [a^r (a^r b)^r\, a^r\,]\, b\\
&= (a\phi_{\alpha,\beta})\, b\,;
\end{aligned}$$

the last equality follows from the observation that $(a^r b)^r$ is an idempotent element in S_β and so C4 also holds.

C5. For any $\alpha \geq \beta$, and any $a \in S_\alpha$, $b \in S_\beta$

$$\begin{aligned}
ba^r &= (ba^r)^{r+1} && \text{(by Lemma 5:1.6 since } ba^r \text{ is regular)}\\
&= (ba^r)\,(ba^r)^{r-1}\,(ba^r)\\
&= (ba^r)\,(ba^r)^{r-1}\,(ba^r a^r) && \text{(since } a^r a^r = a^r \text{ from Lemma 5:1.5)}\\
&= b\,[a^r (ba^r)^r\, a^r\,]\\
&= b(a\phi_{\alpha,\beta})\,;
\end{aligned}$$

we have the last equality by noting that $(ba^r)^r$ is an idempotent element of S_β, and thus C5 also holds.

We have thus proved that every semigroup in $(\mathcal{U}\oplus_{(n,m)}(\mathcal{S}\vee\mathcal{W}))\cap\mathcal{R}^{(\infty)}$ is a dense semilattice of the family $\{S_\alpha: \alpha \in \Gamma\}$ of semigroups, each member of which is contained in $\mathcal{U}\vee\mathcal{W}\subseteq[(x^ry^rx^r)^r = x^r]$. By Theorem 5:1.3, S belongs to the variety $(\mathcal{U}\vee\mathcal{W})\vee\mathcal{S}$ and thus the required equality holds. ❑

Corollary 5:1.7 *With the notation of* Theorem 5:1.4, Lemma 5:1.5, *and* Lemma 5:1.6, *for every semigroup* S *in the class* $(\mathcal{U}\oplus_{(n,m)}(\mathcal{S}\vee\mathcal{W}))\cap\mathcal{R}^{(\infty)}$, *the subsemigroup* Reg(S) *is an orthodox normal band of groups.*

Proof. We know that Reg(S) is a generalised inverse semigroup. We need only prove the existence of a congruence which partitions Reg(S) into a family of pairwise disjoint groups. We know from Corollary 3:5.16 that every normal band is a strong semilattice of rectangular bands. Consider the normal band $E(S) = \bigcup\{E(S_\alpha): \alpha\in\Gamma\}$. Denote by $\{\Phi_{\alpha,\beta}: \alpha \geq \beta,\ \alpha, \beta \in \Gamma\}$ the family of homomorphisms associated with this strong semilattice decomposition, where each $\Phi_{\alpha,\beta}$ is a homomorphism from S_α into S_β. given (as defined in Corollary 3:5.16) by:

$$\Phi_{\alpha,\beta} : E(S_\alpha) \to E(S_\beta),\ x \mapsto\ = xex,\ \text{for any } e \in E(S_\beta).$$

Now, define an equivalence relation Θ on Reg(S) as follows:

$$\Theta = \{(a,b): a^r = b^r,\ a,b \in \text{Reg}(S)\}.$$

We will show that Θ is a congruence. Take any $(a,b) \in \Theta$ and any c in Reg(S). Then $a,b \in \text{Reg}(S_\alpha)$ and $c \in \text{Reg}(S_\beta)$ for some $\alpha,\beta \in \Gamma$. From the proof of Lemma 5:1.5, the elements $(ac)^r$, a^r, and c^r are all idempotents, and so we have that

$$(ac)^r = (ac)(ac)^{r-2}(ac)$$

$$= (a^{r+1}c)\,(ac)^{r-2}\,(ac^{r+1})$$

(from Lemma 5:1.6, $a^{r+1} = a$ and $c^{r+1} = c$)

$$= a^r(ac)^rc^r$$

$$= (a^r\Phi_{\alpha,\alpha\beta})((ac)^r\Phi_{\alpha\beta,\alpha\beta})(c^r\Phi_{\beta,\alpha\beta}) \quad \text{(since E(S) is a strong semilattice)}$$

$$= (a^r\Phi_{\alpha,\alpha\beta})(c^r\Phi_{\beta,\alpha\beta}) \ = a^rc^r.$$

We have the second last equality by noting that since $E(S_{\alpha\beta})$ is a rectangular band, it satisfies the identity $xyz = xz$. Thus $(ac)^r = a^r c^r$. Similarly, $(bc)^r = b^r c^r$. From these equalities, together, we have $(ac)^r = a^r c^r = b^r c^r = (bc)^r$ and so $(ac,bc) \in \Theta$. By the symmetry of the arguments, $(ca,cb) \in \Theta$. Hence Θ is a congruence on Reg(S). Clearly, Θ is idempotent-element-separating, and since every Θ-class contains an idempotent element, each Θ-class is a group. Hence $\mathrm{Reg}(S)/\Theta \cong E(S)$. We have thus shown that Reg(S) is an orthodox normal band of groups. ❑

As shown in Theorem 3:5.14, a semigroup is an orthodox normal band of groups if and only if it is a strong semilattice of rectangular groups. The maps in the proof of Theorem 5:1.4 do not induce a strong semilattice decomposition of Reg(S), but we do see that each $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ is a homomorphism: for any a , $b \in S_\alpha$, and any $g \in E(S_\beta)$,

$$(ab)\phi_{\alpha,\beta} = (ab)^r g(ab)^r = a^r b^r g a^r b^r = (a^r g a^r)(b^r g b^r) = (a\phi_{\alpha,\beta})(b\phi_{\alpha,\beta}).$$

Corollary 5:1.8 *For any nodal subvariety $\mathcal{U}$ of $\mathcal{T}^{(n,m)}$, and the variety $\mathcal{S}$ of all semilattices, the variety*

$$\mathcal{U} \vee \mathcal{S} = (\mathcal{U} \oplus_{(n,m)} \mathcal{S})) \cap \mathcal{R}^{(\infty)}$$

consists of all dense semilattices of structurally trivial semigroups in $\mathcal{U}$.

Proof. We have the statement by putting $\mathcal{W} = \mathcal{T}$ in the statement of Theorem 5:1.4. We observe that by Theorem 5:1.3, a semigroup S belongs to $\mathcal{U} \vee \mathcal{S}$ if and only if it is a dense semilattice of semigroups in $\mathcal{U}$. ❑

In the following result we give an alternative characterisation for semigroups which are both structurally strong semilattices of groups and which are also nilpotent extensions of regular semigroups.

Theorem 5:1.9 *Let S^k be a regular semigroup such that $S/\theta(n,m)$ is a strong semilattice of groups. Then the following relation is a congruence:*

$$\tau_S = \mathcal{H}_S \cap (S^k \times S^k) \cup \{(x,x) : x \in S \setminus S^k\},$$

where $\mathcal{H}_s$ is the Green's relation on S.

Proof. For each $s \in S$, we will denote by s^0 the idempotent element of the group containing the regular element s^k. Take any $a,b \in S^k$ such that $(a,b) \in \mathcal{H}_S = \mathcal{R}_S \cap \mathcal{L}_S$. Then for any element s of S, since $\mathcal{L}_S$ is a right congruence, we have $(as,bs) \in \mathcal{L}_S$. To show that τ_S is a congruence, it suffices to show that $(as,bs) \in \mathcal{R}_S$.

As assumed in the proof of Theorem 5:1.4, the quotient $S/\theta(n,m)$ is a strong semilattice, and so there exists a semilattice Γ and a family of pairwise disjoint groups $\{G_\alpha : \alpha \in \Gamma\}$ such that

$$S/\theta(n,m) = \bigcup \{G_\alpha : \alpha \in \Gamma\}.$$

Now, for each $\alpha \in \Gamma$ define

$$S_\alpha = \{x \in S : x\theta(n,m) \in G_\alpha\},$$

the union of all $\theta(n,m)$-classes of S indexed by elements of the group G_α. Now since $(a,b) \in \mathcal{H}_S$ there exists some $\alpha \in \Gamma$ such that $a,b \in S_\alpha$. Now, since a and b are $\mathcal{R}_S$-related in S, there exist x,y in $S^{(1)}$ such that $a = bx$ and $b = ay$. Then for any $s \in S_\gamma$, $as = bxs \in S_{\alpha\gamma}$ and $b = ays \in S_{\alpha\gamma}$. Let $(s^k)^{-1}$ denote the group inverse of the regular element s^k, and $s^0 = s^k(s^k)^{-1}$.

Now,

$$as = bxs = b(b^0xs)$$

$$= b(b^0)(b^0xs)^0(b^0xs)$$

$$= b\,(b^0)\,(b^0x^0s^0)\,(b^0xs)$$

(the elements $(b^0xs)^0$ and $(b^0x^0s^0)$ are both idempotents in $S_{\alpha\gamma}$ and are therefore $\theta(n,m)$-related in S)

$$= b\,(b^0)\,(s^0x^0b^0)\,(b^0xs)$$

(again, since Γ is a semilattice, the idempotent elements $b^0x^0s^0$ and $s^0x^0b^0$ are both idempotents in $S_{\alpha\gamma}$ and are therefore $\theta(n,m)$-related in S)

$$= b\,(b^0)\,(s^k(s^k)^{-1}x^0b^0)\,(b^0xs)$$

$$= (bs)\,(s^{k-1}(s^k)^{-1}x^0b^0xs).$$

Similarly, by making use of the equality $bs = ays$, one can also show that

$$bs = (as)\,(s^{k-1}(s^k)^{-1}y^0b^0xs).$$

These, together, imply that $(as,bs) \in \mathcal{R}_S$ and hence $(as,bs) \in \mathcal{H}_S$. This proves that τ_S is a right congruence. By dual arguments, one can also show that τ_S is a left congruence on S. ❑

Theorem 5:1.10 *If* S^k *is regular and* $S/\theta(n,m)$ *is a strong semilattice, then* S *is a subdirect product of* $S/\theta(n,m)$ *and* S/τ_s*, where* τ_s *is the congruence defined in Theorem 5:1.9.*

Proof. Take any a,b in S^k such that $(a,b) \in \theta(n,m)$ and $(a,b) \in \tau_s$. Then $a^0 = b^0$ since $(a,b) \in \mathcal{H}_s$. Therefore, $a = a^0aa^0 = b^0bb^0 = b$. Clearly, τ_s separates idempotent elements, relates all elements within each group $\mathcal{H}_s$-classes, and each non-regular element forms a singleton class. Hence S/τ_s is a nilpotent extension of a normal band. We know, by assumption, that $S/\theta(n,m)$ is a strong semilattice of groups. ❑

Corollary 5:1.11 *For any semigroup variety* $\mathcal{W}$ *consisting solely of groups, any nodal subvariety* $\mathcal{U}$ *of* $\mathcal{T}^{(n,m)}$*, and the variety* $\mathcal{S}$ *of all semilattices,*

$$\mathcal{U} \vee \mathcal{S} \vee \mathcal{W} = \{(\mathcal{U} \oplus_{(n,m)} \mathcal{S}) \cap \mathcal{R}^{(\infty)}\} \vee \mathcal{W} = (\mathcal{U} \oplus_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}$$
$$= \{\mathcal{U} \oplus_{(n,m)} \mathcal{W}\} \vee \mathcal{S}$$

Proof. We have by the associativity of the operation $\vee$ of taking varietal joins, and by Corollary 5:1.8 that:

$$\mathcal{U} \vee \mathcal{S} \vee \mathcal{W} = \{\mathcal{U} \vee \mathcal{S}\} \vee \mathcal{W} = \{(\mathcal{U} \oplus_{(n,m)} \mathcal{S}) \cap \mathcal{R}^{(\infty)}\} \vee \mathcal{W}.$$

This proves the first equality in the statement. The second equality is of course due to Theorem 5:1.4:

$$\mathcal{U} \vee (\mathcal{S} \vee \mathcal{W}) = (\mathcal{U} \oplus_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}$$

The third equality is due to the commutativity and associativity of the operation $\vee$, and from Theorem 4:4.9 as $\mathcal{U}$ is a nodal variety:

$$(\mathcal{U} \vee \mathcal{W}) \vee \mathcal{S} = (\mathcal{U} \oplus_{(n,m)} \mathcal{W}) \vee \mathcal{S}. \quad ❑$$

5:2 LATTICES OF STRUCTURALLY INVERSE SEMIGROUP VARIETIES

The lattice $\mathcal{L}(\mathcal{S})$ of all varieties of semilattices has only two distinct subvarieties: namely the variety $\mathcal{S}$ itself, and the trivial variety $\mathcal{T}$. It is for this reason, that we say $\mathcal{S}$ is an atom on the lattice of all semigroup varieties. For those results proved in Chapter 4, if we are able to prove a generalisation involving the variety $\mathcal{S}$, then we will assume that the proved statement holds for all subvarieties of $\mathcal{S}$.

Theorem 5:2.1 *For any nodal variety $\mathcal{U} \subseteq \mathcal{T}^{(n,m)}$ of structurally trivial semigroups, any semigroup variety $\mathcal{W}$ of groups, and any variety $\mathcal{Y}$ of semilattices, the following equalities hold for any $\mathcal{V} \subseteq \mathcal{U} \vee \mathcal{W} \vee \mathcal{S}$*

$$\mathcal{V} = \{(\mathcal{V} \cap \mathcal{U}) \oplus_{(n,m)} [(\mathcal{V} \cap \mathcal{W}) \vee (\mathcal{V} \cap \mathcal{S})]\} \cap \mathcal{R}^{(\infty)}$$

$$= (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}) \vee (\mathcal{V} \cap \mathcal{S}).$$

Proof. Case (i): $\mathcal{V} \cap \mathcal{S} = \mathcal{T}$. The statement reduces exactly to the equalities of Theorem 4:4.9 and so there is nothing further to prove.

Case (ii): $\mathcal{V} \cap \mathcal{S} \neq \mathcal{T}$. This implies that $\mathcal{V} \cap \mathcal{S} = \mathcal{S}$. The problem then reduces to proving the following two equalities:

$$\mathcal{V} = \{(\mathcal{V} \cap \mathcal{U}) \oplus_{(n,m)} [(\mathcal{V} \cap \mathcal{W}) \vee \mathcal{S}]\} \cap \mathcal{R}^{(\infty)} = (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}) \vee \mathcal{S}.$$

Since $\mathcal{V} \cap \mathcal{U}$, $\mathcal{V} \cap \mathcal{W}$, and $\mathcal{V} \cap \mathcal{S} = \mathcal{S}$ are all subvarieties of $\mathcal{V}$, and $\mathcal{V}$ is assumed to be a variety, we have:

$$(\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}) \vee \mathcal{S} \subseteq \mathcal{V}.$$

Now,

$$\mathcal{V} = \mathcal{V} \cap \{\mathcal{U} \vee \mathcal{W} \vee \mathcal{S}\} \qquad (\text{since } \mathcal{V} \subseteq \mathcal{U} \vee \mathcal{W} \vee \mathcal{S})$$

$$= \mathcal{V} \cap \{S : S/\theta(n,m) \in (\mathcal{W} \vee \mathcal{S}),\ e\theta(n,m) \in \mathcal{U} \text{ for all } e^2 = e \in S\} \cap \mathcal{R}^{(\infty)}$$

$$= \{S : S/\theta(n,m) \in \mathcal{V} \cap (\mathcal{W} \vee \mathcal{S}),\ e\theta(n,m) \in \mathcal{V} \cap \mathcal{U} \text{ for all } e^2 = e \in S\} \cap \mathcal{R}^{(\infty)}$$

$$= \{(\mathcal{V} \cap \mathcal{U}) \oplus_{(n,m)} [\mathcal{V} \cap (\mathcal{W} \vee \mathcal{S})]\} \cap \mathcal{R}^{(\infty)}.$$

The second equality above holds by Theorem 5:1.4. The third equality holds since for every $S \in \mathcal{V} \subseteq \mathcal{U} \vee \mathcal{W} \vee \mathcal{S}$, we have by Theorem 5:1.4 that $S/\theta(n,m) \in (\mathcal{W} \vee \mathcal{S})$ and $e\theta(n,m) \in \mathcal{U}$ for all $e^2 = e$. Now, $\mathcal{V}$ being a variety is also closed under taking subsemigroups and homomorphic images and so $S/\theta(n,m) \in \mathcal{V} \cap (\mathcal{W} \vee \mathcal{S})$ and $e\theta(n,m) \in \mathcal{V} \cap \mathcal{U}$ for all $e^2 = e \in S$. To complete the proof, will show that

$$\mathcal{V} = \{(\mathcal{V} \cap \mathcal{U}) \oplus_{(n,m)} [\mathcal{V} \cap (\mathcal{W} \vee \mathcal{S})]\} \cap \mathcal{R}^{(\infty)} \subseteq (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}) \vee \mathcal{S}.$$

But this inclusion follows easily from Theorem 5:1.10 since every semigroup S in the class

$$\mathcal{V} = \{ (\mathcal{V} \cap \mathcal{U}) \oplus_{(n,m)} [\mathcal{V} \cap (\mathcal{W} \vee \mathcal{S})]\} \cap \mathcal{R}^{(\infty)}$$

is a subdirect product of a nilpotent extension of a dense semilattice of semigroups in $\mathcal{V} \cap \mathcal{U}$ (namely S/τ_S) and a strong semilattice of groups in $\mathcal{W}$ (namely $S/\theta(n,m)$). Therefore, the quotient S/τ_S belongs to $(\mathcal{V} \cap \mathcal{U}) \vee \mathcal{S}$ by Corollary 5:1.8, while $S/\theta(n,m)$ belongs to the variety $(\mathcal{V} \cap \mathcal{W}) \vee \mathcal{S}$ by Petrich (1984). Thus we have shown that S belongs to

$$((\mathcal{V} \cap \mathcal{U}) \vee \mathcal{S}) \vee ((\mathcal{V} \cap \mathcal{W}) \vee \mathcal{S}) = (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{W}) \vee \mathcal{S};$$

and that completes the proof. ❑

We point out that, in general, the class $\mathcal{U} \oplus_{(n,m)} (\mathcal{X} \vee \mathcal{W})$ may not form a variety of semigroups. However, the following result shows that each class of the type $\mathcal{U} \oplus_{(n,m)} (\mathcal{X} \vee \mathcal{W})$ is uniquely determined by the triple: $\mathcal{U}, \mathcal{X}, \mathcal{W}$, where $\mathcal{U}$ is not necessarily nodal.

Proposition 5:2.2 *For any structurally trivial varieties $\mathcal{U}$ and $\mathcal{V}$, any varieties $\mathcal{X}$ and $\mathcal{Y}$ of semilattices, and any semigroup varieties $\mathcal{W}$ and $\mathcal{Z}$ formed entirely by groups, if*

$$\mathcal{U} \oplus_{(n,m)} (\mathcal{X} \vee \mathcal{W}) = \mathcal{V} \oplus_{(s,t)} (\mathcal{Y} \vee \mathcal{Z})$$

then $\mathcal{U} = \mathcal{V}$, $\mathcal{X} = \mathcal{Y}$ and $\mathcal{W} = \mathcal{Z}$.

Proof Suppose that $\mathcal{U} \oplus_{(n,m)} (\mathcal{X} \vee \mathcal{W}) = \mathcal{V} \oplus_{(s,t)} (\mathcal{Y} \vee \mathcal{Z})$. Then since

$$\mathcal{U} \subseteq \mathcal{U} \oplus_{(n,m)} (\mathcal{X} \vee \mathcal{W}) = \mathcal{V} \oplus_{(s,t)} (\mathcal{Y} \vee \mathcal{Z}),$$

for every semigroup S in $\mathcal{U}$, the quotient $S/\theta(s,t)$ belongs to $\mathcal{Y} \vee \mathcal{Z}$ and all idempotent $\theta(s,t)$-classes belong to $\mathcal{V}$. But S, being structurally trivial, forms an entire idempotent $\theta(s,t)$-class. Hence $S \in \mathcal{V}$, proving that $\mathcal{U} \subseteq \mathcal{V}$. Similarly, $\mathcal{V} \subseteq \mathcal{U}$. These containments, together, imply that $\mathcal{V} = \mathcal{U}$.
Now, we also have

$$\mathcal{W} \subseteq \mathcal{U}\oplus_{(n,m)}(\mathcal{X} \vee \mathcal{W}) = \mathcal{V}\oplus_{(s,t)}(\mathcal{Y} \vee \mathcal{Z}).$$

This means that for every semigroup S in $\mathcal{W}$, the quotient $S/\theta(s,t)$ belongs to $\mathcal{Y} \vee \mathcal{Z}$ and all idempotent $\theta(s,t)$-classes belong to $\mathcal{V}$. But S being a group is reductive and so we must have $S = S/\theta(s,t) \in \mathcal{Y} \vee \mathcal{Z}$. Then from Petrich (1984) we must have $S = S/\theta(s,t) \in \mathcal{Z}$ since S is a group. We have thus shown that $\mathcal{W} \subseteq \mathcal{Z}$. Similarly, $\mathcal{Z} \subseteq \mathcal{W}$. These containments, together, imply that $\mathcal{W} = \mathcal{Z}$.

Finally, to prove that $\mathcal{X} = \mathcal{Y}$, take any semigroup S in the class

$$\mathcal{X} \subseteq \mathcal{U}\oplus_{(n,m)} (\mathcal{X} \vee \mathcal{W}) = \mathcal{V}\oplus_{(s,t)} (\mathcal{Y} \vee \mathcal{Z}).$$

This means that $S/\theta(s,t)$ belongs to $\mathcal{Y} \vee \mathcal{Z}$ and all idempotent $\theta(s,t)$-classes belong to $\mathcal{V}$. But S being a semilattice is reductive and so $S = S/\theta(s,t) \in \mathcal{Y} \vee \mathcal{Z}$. But then from Petrich (1984) this means that since S is a semilattice we have $S = S/\theta(s,t) \in \mathcal{Y}$. This proves that $\mathcal{X} \subseteq \mathcal{Y}$. Similarly, $\mathcal{Y} \subseteq \mathcal{X}$. These containments, together, imply that $\mathcal{X} = \mathcal{Y}$. ❑

Theorem 5:2.3 *Fix any nodal subvariety $\mathcal{U}$ of $\mathcal{T}^{(n,m)}$ and fix any semigroup variety $\mathcal{Y}$ of strong semilattices of groups.*
Then the following lattice map is an isomorphism:

$$\Phi: \mathcal{L}(\mathcal{Y}) \to [\mathcal{U}, \mathcal{U} \vee \mathcal{Y}], \quad \mathcal{X} \mapsto \mathcal{U} \vee \mathcal{X}, \qquad (\mathcal{X} \subseteq \mathcal{Y}).$$

Hence, the lattice $\mathcal{L}(\mathcal{Y})$ of all subvarieties of $\mathcal{Y}$ is isomorphic to the lattice interval $[\mathcal{U}, \mathcal{U} \vee \mathcal{Y}]$.

Proof. Case (i): Suppose that $\mathcal{Y}$ consists entirely of groups. Then the above statement reduces exactly to Lemma 4:4.13 and so there is nothing further to prove.

Case (ii) : Suppose that $\mathcal{Y} = \mathcal{W} \vee \mathcal{S}$, where $\mathcal{S}$ is the variety of semilattices and $\mathcal{W}$ is a variety formed entirely of groups. To show that Φ is onto, take any variety $\mathcal{V}$ in the interval $[\mathcal{U}, \mathcal{U} \vee \mathcal{Y}]$. Then this means that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{U} \vee \mathcal{Y}$. Now,

$$\begin{aligned} \mathcal{V} &= (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{S}) \vee (\mathcal{V} \cap \mathcal{W}) && \text{(from Lemma 5:2.1)} \\ &= \mathcal{U} \vee ((\mathcal{V} \cap \mathcal{S}) \vee (\mathcal{V} \cap \mathcal{W})) && \text{(since } \mathcal{U} \subseteq \mathcal{V}) \\ &= ((\mathcal{V} \cap \mathcal{S}) \vee (\mathcal{V} \cap \mathcal{W}))\Phi; && \text{(since } (\mathcal{V} \cap \mathcal{S}) \vee (\mathcal{V} \cap \mathcal{W}) \subseteq \mathcal{Y}) \end{aligned}$$

and so Φ is onto. We will show next that Φ preserves varietal joins. Take any subvarieties $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathcal{Y}$. Then

$$(\mathcal{X}_1 \vee \mathcal{X}_2)\Phi = \mathcal{U} \vee (\mathcal{X}_1 \vee \mathcal{X}_2) = (\mathcal{U} \vee \mathcal{X}_1) \vee (\mathcal{U} \vee \mathcal{X}_2) = (\mathcal{X}_1\Phi) \vee (\mathcal{X}_2\Phi).$$

We will show next that Φ is one-to-one. Suppose that there exist distinct varieties of strong semilattices of groups say $\mathcal{W}_1 \vee \mathcal{S} \neq \mathcal{W}_2 \vee \mathcal{S}$ such that $\mathcal{U} \vee \mathcal{W}_1 \vee \mathcal{S} = \mathcal{U} \vee \mathcal{W}_2 \vee \mathcal{S}$. Then without loss of generality there exists $\Gamma \in \mathcal{W}_1 \vee \mathcal{S}$ such that $\Gamma \notin \mathcal{W}_2 \vee \mathcal{S}$. For any non trivial member A of $\mathcal{U}$, consider the direct product $S = A \times \Gamma$. Then by Lemma 1:1.9 we have that $S/\theta(n,m)$ is isomorphic to Γ since Γ is reductive and $A/\theta(n,m)$ is trivial; and every idempotent $\theta(n,m)$-class of S is isomorphic to A. Clearly, S belongs to $\mathcal{U} \vee \mathcal{W}_1 \vee \mathcal{S}$ but since $\Gamma \notin \mathcal{W}_2 \vee \mathcal{S}$, the semigroup S does not belong to $\mathcal{U} \vee \mathcal{W}_2 \vee \mathcal{S}$. By this contradiction, we must have have $\mathcal{W}_1 \vee \mathcal{S} = \mathcal{W}_2 \vee \mathcal{S}$ and that completes the proof that Φ is an isomorphism. ❑

Recall that a structurally trivial semigroup variety $\mathcal{U}$ is called nodal if there exists some (n,m) such that for any variety $\mathcal{W}$ of groups $\mathcal{U} \oplus_{(n,m)} \mathcal{W} = \mathcal{U} \otimes_{(n,m)} \mathcal{W} = \mathcal{U} \vee \mathcal{W}$; and we denoted by $\mathcal{L}_N(\mathcal{U})$ the lattice of all nodal subvarieties of $\mathcal{U}$. In order to generalise Theorem 4:4.12 consider $\mathcal{L}_N(\mathcal{U} \vee \mathcal{Z})$, the lattice structure of the partially ordered set:

$$\{\mathcal{A} \vee \mathcal{B} : \mathcal{A} \in \mathcal{L}_N(\mathcal{V}),\ \mathcal{B} \in \mathcal{L}(\mathcal{Z})\},$$

where $\mathcal{Z}$ is a variety of strong semilattices of groups.

Theorem 5:2.4 *For any semigroup variety $\mathcal{W}$ consisting entirely of groups, and any nodal subvariety $\mathcal{U}$ of $\mathcal{T}^{(n,m)}$, the following lattice map is an isomorphism:*

$$\Psi : \mathcal{L}_N(\mathcal{U} \vee \mathcal{S} \vee \mathcal{W}) \to \mathcal{L}_N(\mathcal{U}) \times \mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{W}),$$

$$\mathcal{V} \mapsto (\mathcal{V} \cap \mathcal{U}, \mathcal{V} \cap \mathcal{S}, \mathcal{V} \cap \mathcal{W}).$$

The reverse map sends $(\mathcal{A},\mathcal{B},\mathcal{C})$ to $\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}$. Hence, the lattice structure of $\mathcal{L}_N(\mathcal{U} \vee \mathcal{S} \vee \mathcal{W})$ is isomorphic to the lattice direct product

$$\mathcal{L}_N(\mathcal{U}) \times \mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{W}).$$

Proof. The map Ψ clearly preserves arbitrary varietal meets. To show that it is one to one, take any subvarieties $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{U} \vee \mathcal{S} \vee \mathcal{W}$ such that $\mathcal{V}_1\Psi = \mathcal{V}_2\Psi$. This implies that:

$$\mathcal{V}_1 \cap \mathcal{U} = \mathcal{V}_2 \cap \mathcal{U}, \qquad \mathcal{V}_1 \cap \mathcal{S} = \mathcal{V}_2 \cap \mathcal{S} \qquad \text{and} \qquad \mathcal{V}_1 \cap \mathcal{W} = \mathcal{V}_2 \cap \mathcal{W}.$$

From Lemma 5:2.1,

$$\mathcal{V}_1 = (\mathcal{U} \cap \mathcal{V}_1) \vee (\mathcal{V}_1 \cap \mathcal{S}) \vee (\mathcal{V}_1 \cap \mathcal{W}) = (\mathcal{U} \cap \mathcal{V}_2) \vee (\mathcal{V}_2 \cap \mathcal{S}) \vee (\mathcal{V}_2 \cap \mathcal{W}) = \mathcal{V}_2.$$

To show that Ψ is an isomorphism, it suffices to prove that it is a bijection (in view of Lemma 0:4.20). To prove that Ψ is onto, we will show that for any element $(\mathcal{A},\mathcal{B},\mathcal{C})$ of $\mathcal{L}(\mathcal{U}) \times \mathcal{L}(\mathcal{S}) \times \mathcal{L}(\mathcal{W})$, $(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C})\Psi = (\mathcal{A},\mathcal{B},\mathcal{C})$. This is equivalent to proving each of the following equalities:

(i) $(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{U} = \mathcal{A}$

(ii) $(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{S} = \mathcal{B}$

(iii) $(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{W} = \mathcal{C}$.

Equality (i): Since $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{A} \vee \mathcal{B} \vee \mathcal{C}$, we have $\mathcal{A} \subseteq (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{U}$. Conversely, take any semigroup S in $(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{U}$. By Theorem 4:1.4, $S/\theta(n,m)$ belongs to $\mathcal{B} \vee \mathcal{C}$ and each idempotent $\theta(n,m)$-class belongs to $\mathcal{A}$. But S being in $\mathcal{U}$ is a structurally trivial semigroup and so $S/\theta(n,m)$ is a trivial semigroup. This means that S forms an entire idempotent $\theta(n,m)$-class, contained in $\mathcal{A}$. This proves that $\mathcal{A} \supseteq (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{U}$ and so the equality holds.

Equality (ii): Since $\mathcal{B} \subseteq \mathcal{S}$ and $\mathcal{B} \subseteq \mathcal{A} \vee \mathcal{B} \vee \mathcal{C}$, we have $\mathcal{B} \subseteq (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{U}$. Conversely, take any semigroup S in $(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{S}$. By Theorem 4:1.4, $S/\theta(n,m)$ belongs to $\mathcal{B} \vee \mathcal{C}$ and each idempotent $\theta(n,m)$-class belongs to $\mathcal{A}$. But S being in $\mathcal{S}$ is a semilattice and so is reductive. This means that $\theta(n,m)$ is the identity relation on S. Hence $S = S/\theta(n,m) \in \mathcal{B} \vee \mathcal{C}$. From Petrich (1984) since S is group we must have $S = S/\theta(n,m) \in \mathcal{B}$. Thus we have shown that $\mathcal{B} \supseteq (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{U}$ and so the equality holds.

Equality (iii): Since $\mathcal{C} \subseteq \mathcal{W}$ and $\mathcal{C} \subseteq \mathcal{A} \vee \mathcal{B} \vee \mathcal{C}$, we have $\mathcal{C} \subseteq (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{W}$. Conversely, take any semigroup S in $(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{W}$. By Theorem 4:1.4, $S/\theta(n,m)$ belongs to $\mathcal{B} \vee \mathcal{C}$ and each idempotent $\theta(n,m)$-class belongs to $\mathcal{A}$.

But S being in $\mathcal{W}$ is a group and so is reductive. This means that $\theta(n,m)$ is the identity relation on S. Hence $S = S/\theta(n,m) \in \mathcal{B} \vee \mathcal{C}$. From Petrich (1984), since S is semilattice $S = S/\theta(n,m) \in \mathcal{C}$. Thus we have shown that $\mathcal{C} \supseteq (\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \cap \mathcal{W}$ and so the equality holds.

We have thus shown that Ψ is onto. Let Ψ^{-1} denote the reverse map of Ψ, and take any subvariety $\mathcal{V}$ of $\mathcal{U} \vee \mathcal{S} \vee \mathcal{W}$.
Then,
$(\mathcal{V}\Psi)\Psi^{-1} = ((\mathcal{V} \cap \mathcal{U}, (\mathcal{V} \cap \mathcal{S}), (\mathcal{V} \cap \mathcal{W}))\Psi^{-1} = (\mathcal{V} \cap \mathcal{U}) \vee (\mathcal{V} \cap \mathcal{S}) \vee (\mathcal{V} \cap \mathcal{W}) = \mathcal{V}$.
We have the last equality from Lemma 5:2.1. ❑

Corollary 5:2.5 *With the notations of Theorem 5:2.4 we have that for any* $\mathcal{V} \subseteq \mathcal{T}^{(n,m)} \vee \mathcal{S} \vee \mathcal{W}$,

$$\mathcal{L}_N(\mathcal{V}) \cong \mathcal{L}_N(\mathcal{V} \cap \mathcal{T}^{(n,m)}) \times \mathcal{L}(\mathcal{V} \cap \mathcal{S}) \times \mathcal{L}(\mathcal{V} \cap \mathcal{W})$$

and in particular,

$$\mathcal{L}_N(\mathcal{T}^{(n,m)} \vee \mathcal{S} \vee \mathcal{W}) \cong \mathcal{L}_N(\mathcal{T}^{(n,m)}) \times \mathcal{L}(\mathcal{S}) \times (\mathcal{W}). \quad ❑$$

Within the context of existence varieties, Hall (1991) proved a result which resembles Theorem 5:2.4. In fact he proved that the lattice $\mathcal{L}_{ev}(\mathcal{V} \vee \mathcal{W})$ of all sub e-varieties of the join of any e-variety $\mathcal{V}$ of rectangular bands and any e-variety $\mathcal{W}$ of inverse semigroups, is isomorphic to the direct product $\mathcal{L}_{ev}(\mathcal{V}) \times \mathcal{L}_{ev}(\mathcal{W})$. Earlier, Petrich and Reilly (1984) described all joins of strict inverse semigroups and rectangular band varieties within the context of unary semigroup varieties.

5:3 OPEN PROBLEMS

Problem 5:3.1 Let $\mathcal{W}$ be a semigroup variety consisting entirely of groups, and $\mathcal{U}$ be any nodal subvariety of $\mathcal{T}^{(n,m)}$. The containment $\mathcal{U} \vee \mathcal{S} \vee \mathcal{W} \subset \mathcal{U}\otimes_{(n,m)}(\mathcal{S} \vee \mathcal{W})$ is strict in general. We have attempted to describe the lattice structure of $\mathcal{L}_N(\mathcal{U} \vee \mathcal{S} \vee \mathcal{W})$ in this chapter, but nothing is known about the structure of the larger lattice $\mathcal{L}(\mathcal{U}\otimes_{(n,m)}(\mathcal{S} \vee \mathcal{W}))$. ❑

Problem 5:3.2 Petrich (1975) proved that every semigroup variety $\mathcal{V}$ of orthodox bands of groups can be expressed as a join $\mathcal{V} = (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G})$, where $\mathcal{B}$ is the variety of all bands and $\mathcal{G}$ is the class of all groups (see Lemma 0:4.23). Moreover, he also showed that the lattice $\mathcal{L}(\mathcal{V})$ is isomorphic to the direct product $\mathcal{L}(\mathcal{V} \cap \mathcal{B}) \times \mathcal{L}(\mathcal{B} \cap \mathcal{G})$ (see (0:4.24)). This suggests the following problems.

(i) Describe the lattice $\mathcal{L}(\mathcal{V}^{(n,m)})$ for any variety $\mathcal{V}$ of orthodox bands of groups. Can the results of this chapter be generalised any further to these?

(ii) Describe the lattice structure of $\mathcal{L}(\mathcal{V}^{(n,m)})$ for any variety $\mathcal{V}$ for which the lattice $\mathcal{L}(\mathcal{V})$ is known. For example, let $\mathcal{V}$ be the variety of all bands or let $\mathcal{V}$ be a Burnside variety of regular semigroups etc. Certain special cases of this are considered in the next two chapters.

(iii) Let $\mathcal{U}$, $\mathcal{S}$, and $\mathcal{W}$ be as in Theorem 5:1.4. Find a method of obtaining identities which determine $\mathcal{U} \vee \mathcal{S} \vee \mathcal{W}$ if those determining $\mathcal{U}$ and $\mathcal{V}$ are known. Of course, we have $\mathcal{S} = [x^2=x, xy = yx]$. Petrich (1984) gave a way of obtaining (unary) semigroup identities that determine $\mathcal{S} \vee \mathcal{W}$ from those that determine $\mathcal{W}$; and in Petrich (1974) the identities that determine $\mathcal{U}\vee\mathcal{S}\vee\mathcal{W}$ were found for all $\mathcal{U} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^2$ and where $\mathcal{W}$ is a Burnside variety of groups whose indices divide a fixed positive integer. ❑

The following result may be useful in approaching the above problems.

Theorem 5:3.3 *Let $\mathcal{U} \subseteq \mathcal{T}^{(n,m)}$ and $\mathcal{W} = [p_\alpha = q_\alpha]$ be a semigroup variety consisting entirely of groups.*
Then

$$\mathcal{U} \vee \mathcal{S} \vee \mathcal{W} = \{\, S: S/\theta(n,m) \in \mathcal{W} \textit{ and } e\theta(n,m) \in \mathcal{U}, \textit{for all } e^2 = e \in E(S) \,\} \cap \mathcal{R}^{(\infty)}$$

$$= [\,(p_\alpha , q_\alpha) \in \theta(n,m)] \cap \{S : E_{(n,m)}(S) \in \mathcal{S} \vee \mathcal{U}\} \cap \mathcal{R}^{(\infty)}$$

$$= \{S: S \textit{ is a dense semilattice of semigroups in } \mathcal{U} \vee \mathcal{W}\}$$

Proof. Consider the first equality. By definition, the class of semigroups given by $\{S: S/\theta(n,m) \in \mathcal{W}$ and $e\theta(n,m) \in \mathcal{U}$, for all $e^2 = e \in E(S)\} \cap \mathcal{R}^{(\infty)}$ is just the variety $(\mathcal{U} \otimes_{(n,m)} (\mathcal{S} \vee \mathcal{W})) \cap \mathcal{R}^{(\infty)}$. Hence, by Theorem 5:1.4, we have the first equality.

To prove the second equality, take any semigroup S in $\mathcal{U} \vee \mathcal{S} \vee \mathcal{W}$. Then by the first equality, $S/\theta(n,m)$ belongs to $[p_\alpha = q_\alpha] = \mathcal{W}$ and so by Theorem 2:2.6 the semigroup S belongs to $[(p_\alpha , q_\alpha) \in \theta(n,m)]$.
By Corollary 5:1.8

$$E_{(n,m)} = \bigcup \{E_{(n,m)}(S_\alpha): \alpha \in \Gamma\},$$

is a dense semilattice of semigroups in $\{E_{(n,m)}(S_\alpha): \alpha \in \Gamma\}$. By assumption each $E_{(n,m)}(S_\alpha)$ is the unique idempotent $\theta(n,m)$-class contained in S_α, and so is contained in $\mathcal{U}$. Hence by Theorem 5:1.3, $E_{(n,m)}$ belongs to $\mathcal{U} \vee \mathcal{S}$. Thirdly, S belongs to $\mathcal{R}^{(\infty)}$ by Theorem 5:1.4. Thus we have shown that the second class is contained in the third. To prove the reverse containment, take any semigroup S in the third class. Then since $S \in [(p_\alpha , q_\alpha) \in \theta(n,m)]$, by assumption, we have by Theorem 2:2.6 that $S/\theta(n,m)$ belongs to $[p_\alpha = q_\alpha] = \mathcal{W}$. Now, $E_{(n,m)}(S) \in \mathcal{S} \vee \mathcal{U}$, also by assumption, implies that $e\theta(n,m) \in \mathcal{U}$ for every idempotent e. Hence the second equality holds. Consider now the final equality. Trivially, the fourth class is contained in the first. Conversely, the first class is contained in the fourth by Theorem 5:1.4. Hence, all three of the equalities hold, and that completes the proof. ❑

Problem 5:3.4 *Characterise all nilpotent extensions of generalised inverse (or more generally, orthodox or regular) semigroups which are not structurally regular.* ❑

Problem 5:3.5 *Is it true that every nilpotent extension of an orthodox band of groups is also a structurally regular semigroup? Of course the converse is not true (see Example 3:1.14 or Example 3:1.8).* ❑

Problem 5:3.6 *We have shown that* $\mathcal{U} \vee \mathcal{V} \vee \mathcal{S}$ *consists of dense semilattices of groups in* $\mathcal{U} \vee \mathcal{W}$. *We showed in Corollary 5:1.7 that these semigroups are also nilpotent extensions of orthodox normal bands of groups in* $\mathcal{W}$. *Is the converse also true? That is, is every nilpotent extension of an orthodox normal band of groups (from* $\mathcal{W}$*) both a structurally inverse semigroup and a nilpotent extension of regular semigroups?*

CHAPTER 6

VARIETIES OF INFLATED REGULAR SEMIGROUPS

It is shown in this chapter that every semigroup variety $\mathcal{V}$ consisting entirely of n-inflations of regular semigroups can be expressed as a join $\mathcal{V} = \mathcal{W} \vee \mathcal{N}_n$, where $\mathcal{W}$ is a semigroup variety of regular semigroups, and $\mathcal{N}_n$ is a variety of n-nilpotent semigroups. We describe how one can obtain the identities which determine $\mathcal{W} \vee \mathcal{N}_n$ from those identities which determine $\mathcal{W}$.

6:1 N-INFLATED REGULAR SEMIGROUPS

The concept of *n-inflation* is a generalisation by Stojan Bogdanovic and Svetozar Milic (1987) of the concept of *inflation* by Clifford and Preston (1962). Note however, that our usage of the phrase 'n-inflation' is really (n-1)-inflation in the sense of Bogdanovic and Milic. We have chosen this convention to be consistent with our definition of *n-nilpotent* semigroups — those semigroups for which $S^n = \{0\}$. In our notation, a 1-nilpotent extension is an ideal extension by a trivial semigroup (which of course is meaningless since $S^1 = S$), while a 2-nilpotent extension is an ideal extension by a null semigroup.

By a *variety of regular [orthodox] semigroups,* we mean a semigroup variety in which every member is a regular [orthodox] semigroup. For any variety $\mathcal{V}$ of regular semigroups, we shall denote by $\mathcal{V}^{\inf(n)}$ the class of all n-inflations of semigroups in $\mathcal{V}$. It turns out that in the case where $\mathcal{V}$ consists of regular semigroups, $\mathcal{V}^{\inf(n)}$ also forms a semigroup variety. By an *n-inflated variety* $\mathcal{W}$ of regular [orthodox] semigroups, we mean a variety $\mathcal{W}$ for which there exists some variety $\mathcal{V}$ of regular [orthodox] semigroups such that $\mathcal{W} = \mathcal{V}^{\inf(n)}$. In contrast, by a *variety of n-inflated regular [orthodox] semigroups,* we mean a subvariety of a variety of the form $\mathcal{V}^{\inf(n)}$, where $\mathcal{V}$ consists of regular [orthodox] semigroups.

If S is an n-inflation of a regular semigroup, then, as shown in Theorem 3:1.6 the congruence

$$\delta_{n-1} = \theta(n\text{-}1,0) \cap \theta(0,n\text{-}1)$$

turns out to be the kernel of the retract endomorphism ϕ responsible for this n-inflation. As a result of this alternative characterisation of kerϕ, we are able to prove some nice results concerning such semigroups. The reader may wish to refer back to Theorem 3:1.6 where we presented a characterisation of n-inflations of regular semigroups, and of course the congruence δ_{n-1} was first introduced there.

6:2 N-INFLATED VARIETIES OF REGULAR SEMIGROUPS

In this section we give determining identities for varieties of the form $\mathcal{V}^{\mathrm{Inf}(n)}$, where $\mathcal{V}$ is a variety of [orthodox] regular semigroups whose determining identities are known.

Theorem 6:2.1 *Let* $\mathcal{V} = [p_\alpha = q_\alpha]$ *be a semigroup variety consisting entirely of regular semigroups. Then the variety* $\mathcal{V}^{\mathrm{Inf}(n)}$ *of all n-inflations of semigroups in* $\mathcal{V}$ *is determined by the identities*

(6:2.2) $\quad (x_1x_2 \ldots x_{n-1})p_\alpha = (x_1x_2 \ldots x_{n-1})q_\alpha\ , \quad p_\alpha(x_1x_2 \ldots x_{n-1}) = q_\alpha(x_1x_2 \ldots x_{n-1})\ ,$

where the letters $x_1, \ldots, x_{n-1}$ *do not occur in any of the words* p_α *and* q_α, *and* $i \neq j$ *implies* $x_i \neq x_j$.

Proof. Suppose that S satisfies the identities in (6:2.2). Then by Theorem 2:2.6, both the quotients $S/\theta(n\text{-}1,0)$ and $S/\theta(0,n\text{-}1)$ belong to $\mathcal{V}$, and so are both regular. Hence, by Lemma 3:1.5, Reg(S) forms a regular subsemigroup of S. Then by Theorem 3:1.6, S is an n-inflation of Reg(S). Thus to show that S belongs to $\mathcal{V}^{\mathrm{Inf}(n)}$, it suffices to prove that Reg(S) belongs to $\mathcal{V}$. The congruence $\delta_{n-1} = \theta(n\text{-}1,0) \cap \theta(0,n\text{-}1)$ is regular-element-separating (as was shown in the proof of Theorem 3:1.6). Suppose that the words p_α and q_α are formed entirely by elements of Reg(S). Then directly from (6:2.2), we have that

$$(p_\alpha, q_\alpha) \in \delta_{n-1} \cap (\mathrm{Reg}(S) \times \mathrm{Reg}(S)).$$

By the regular element separating nature of δ_{n-1}, we have $p_\alpha = q_\alpha$. Thus we have proved that Reg(S) satisfies the identity $p_\alpha = q_\alpha$, showing that S is an n-inflation of a semigroup in $\mathcal{V}$.

Conversely, suppose that S is an n-inflation of a semigroup in $\mathcal{V}$. Then there exists a retractive endomorphism ϕ from S onto Sϕ, a semigroup in $\mathcal{V}$, such that $S^n \subseteq S\phi$ and that for every element x in Sϕ, $x\phi = x$. For any identity

$p_\alpha = q_\alpha$ defining $\mathcal{V}$, let $\mathbf{p}_\alpha$ and $\mathbf{q}_\alpha$ be the elements of S obtained by substituting elements of S for the variables forming the identity $p_\alpha = q_\alpha$. We observe that since $(\mathbf{p}_\alpha\phi)$ and $(\mathbf{p}_\alpha\phi)$ belong to $S\phi$, and since $S\phi \in \mathcal{V}$, the equality $(\mathbf{p}_\alpha\phi) = (\mathbf{q}_\alpha\phi)$ holds. Now, for any choice of elements $x_1, \ldots, x_{n-1}$ in S, both the elements $(x_1x_2x_3 \ldots x_{n-1})\mathbf{p}_\alpha$ and $(x_1x_2x_3 \ldots x_{n-1})\mathbf{p}_\alpha$ are contained in $S\phi$. Therefore,

$$\begin{aligned}(x_1x_2x_3 \ldots x_{n-1})\mathbf{p}_\alpha &= ((x_1x_2x_3 \ldots x_{n-1})\mathbf{p}_\alpha)\phi \\ &= (x_1\phi)(x_2\phi)(x_3\phi) \ldots (x_{n-1}\phi)(\mathbf{p}_\alpha\phi) \\ &= (x_1\phi)(x_2\phi)(x_3\phi) \ldots (x_{n-1}\phi)(\mathbf{q}_\alpha\phi) \\ &= ((x_1x_2x_3 \ldots x_{n-1})\mathbf{q}_\alpha)\phi \\ &= (x_1x_2x_3 \ldots x_{n-1})\mathbf{q}_\alpha .\end{aligned}$$

One can similarly show that $\mathbf{p}_\alpha(x_1x_2x_3 \ldots x_{n-1}) = \mathbf{q}_\alpha(x_1x_2x_3 \ldots x_{n-1})$. Hence, the variety $\mathcal{V}^{\inf(n)}$ is determined by the identities in (6:2.2). ❑

In the following result, $\mathcal{B}_{(1,1)} = [x^2 = x]$ denotes the variety of all bands.

Corollary 6:2.3 *The variety* $\mathcal{B}_{(1,1)}{}^{\inf(n)}$ *of all n-inflations of bands is determined by the pair of identities*

(6:2.4) $\quad (x_1x_2 \ldots x_{n-1})z^2 = (x_1x_2 \ldots x_{n-1})z$ *and* $\quad z^2(x_1x_2 \ldots x_{n-1}) = z(x_1x_2 \ldots x_{n-1})$.

In particular, if $\mathcal{V} = [x^2 = x, p = q]$ *is a variety of bands, then the variety* $\mathcal{V}^{\inf(n)}$ *of all n-inflations of bands in* $\mathcal{V}$ *is determined within* $\mathcal{B}_{(1,1)}{}^{\inf(n)}$ *by the pair of identities*

(6:2.5) $\quad (x_1x_2 \ldots x_{n-1})p = (x_1x_2 \ldots x_{n-1})q\ ,\ p(x_1x_2 \ldots x_{n-1}) = q(x_1x_2 \ldots x_{n-1})$,

where the letters $x_1, \ldots, x_{n-1}$ *do not occur in any of the words* p *or* q, *and* $x_i \neq x_j$ *if* $i \neq j$. ❑

Given the determining identities for a variety $\mathcal{V}$ of regular semigroups, we now have a way of obtaining the identities that determine $\mathcal{V}^{\inf(n)}$, thanks to Theorem 6:2.1. For a quick example, consider the variety $\mathcal{S}^{\inf(4)}$ of all 4-inflations of semilattices. It is determined by the four identities given below:

$$x(zyk) = x^2(zyk), \quad (zyk)x = (zyk)x^2, \quad (pqr)xy = (pqr)yx, \quad xy(pqr) = yx(pqr).$$

And the variety $\mathcal{A}_n{}^{\inf(5)}$ of all 5-inflations of groups in $\mathcal{A}_n = [xy = yx, x^{n+1} = x]$, is determined by the following set of four identities:

$$\begin{aligned} xy(pqrs) &= yx(pqrs), & (pqrs)xy &= (pqrs)yx, \\ (pqrs)x^{n+1} &= (pqrs)x, & x^{n+1}(pqrs) &= x(pqrs). \end{aligned}$$

In the following result, let $\mathcal{N}_n$ denote the variety of all n-nilpotent semigroups.

Theorem 6:2.6 *For every* $n \geq 1$, $\mathcal{T}^{\inf(n)} = \mathcal{N}_n$, *where* $\mathcal{T}$ *is the trivial variety. Moreover, for any variety* $\mathcal{V}$ *of regular semigroups,*

$$\mathcal{V} \vee \mathcal{N}_n = \mathcal{V}^{\inf(n)} = \mathcal{V}^{(n-1,0)} \cap \mathcal{V}^{(0,n-1)}.$$

Proof. Since the variety $\mathcal{T}$ of trivial semigroups is determined by the identity $z = y$, we have by Theorem 6:2.1 that the variety $\mathcal{T}^{\inf(n)}$ of all n-inflations of trivial semigroups is determined by the pair of identities

(6:2.7) $\quad (x_1 \ldots x_{n-1})z = (x_1 \ldots x_{n-1})y$ and $z(x_1 \ldots x_{n-1}) = y(x_1 \ldots x_{n-1})$.

Take any semigroup S in $\mathcal{T}^{\inf(n)}$ and any elements $s = s_1\, s_2 \ldots s_n$ and $t = t_1\, t_2 \ldots t_{n-1}\, t_n$ of S. Then we have by (6:2.7) that

$$s = s_1\, s_2 \ldots s_{n-1}\, s_n = (s_1\, s_2 \ldots s_{n-1})\, s_n = (s_1\, s_2 \ldots s_{n-1})(t_1\, t_2 \ldots t_{n-1}t_n)$$
$$= (s_1\, s_2 \ldots s_{n-1}t_1)(t_2 \ldots t_{n-1}t_n) = (t_1)(t_2 \ldots t_{n-1}t_n) = t;$$

and so S is n-nilpotent. Conversely, take any n-nilpotent semigroup S. Then since $S^n = \{0\}$, one can show that for all $x \in S^{n-1}$ and any $a,b \in S$, $xa = 0 = xb$ and $ax = 0 = bx$. Hence S satisfies (6:2.7). Then by Theorem 6:2.1 S is an n-inflation of a trivial semigroup and is therefore contained in $\mathcal{T}^{\inf(n)}$. Consider now the first of the remaining set of two equalities. For any semigroup S in $\mathcal{V}^{\inf(n)}$, there exists a retractive endomorphism $\phi : S \to S\phi$ such that $S^n \subseteq S\phi$ and for all x in $S\phi$, $x\phi = x$. Since $S\phi$ is regular, $S^n = S\phi$. Denote the Rees congruence on S with respect to the ideal $S^n = S\phi$ by

$$\delta = \{(x,y): x,y \in S\phi\} \cup \{(x,x): x \in S \backslash S\phi\}, \quad \text{and} \quad \mathrm{Ker}\phi = \{(x,y): x\phi = y\phi\}.$$

Take any $(a,b) \in \delta \cap \mathrm{Ker}\phi$. Then by the definition of δ, either $(a,b) \in S\phi \times S\phi$ or $(a,b) \in (S\backslash S\phi) \times (S\backslash S\phi)$. In the first case, we have $a = b$ since $\mathrm{Ker}\phi$ separates the elements of $S\phi$. In the second case, since δ separates the elements of $S\backslash S\phi$, we have $a = b$. Thus $\delta \cap \mathrm{Ker}\phi$ is the identity relation on S, and so it follows that S is a subdirect product of $S/\mathrm{Ker}\phi$ (a member of $\mathcal{V}$) and S/δ (a member of $\mathcal{N}_k$). Therefore, S belongs to $\mathcal{V} \vee \mathcal{N}_n$. Conversely, from what we have proved earlier, $\mathcal{N}_n = \mathcal{T}^{\inf(n)} \subseteq \mathcal{V}^{\inf(n)}$; and clearly, $\mathcal{V} \subseteq \mathcal{V}^{\inf(n)}$. In view of Theorem 6:2.1, we have by the Birkhoff Theorem (see Evans (1971)) that the class $\mathcal{V}^{\inf(n)}$ forms a semigroup variety, since it is an equational class. The above two containments, together, imply that $\mathcal{V} \vee \mathcal{N}_n \subseteq \mathcal{V}^{\inf(n)}$, and hence the equality $\mathcal{V} \vee \mathcal{N}_n = \mathcal{V}^{\inf(n)}$ holds. Consider now the remaining equality. We have, by

Theorem 2:2.6, by Theorem 6:2.1, and by what we have just proved, that the variety $\mathcal{V}^{(n-1,0)} \cap \mathcal{V}^{(0,n-1)}$ is determined by the same set of identities which determine $\mathcal{V}^{\inf(n)}$, and that proves the remaining equality. ❑

Clarke (1981) gave an alternative set of determining identities for varieties consisting entirely of 2-inflations of completely regular semigroups. To show that these two methods produce different but equivalent sets of determining identities for 2-inflated varieties of completely regular semigroups, consider the variety of all 2-inflations of bands. Clarke's method gives the single identity $xy = x^2y^2$. Our method yields the pair of identities $zx^2 = zx$ and $x^2z = xz$. Clearly, the conjunction of our two identities is equivalent to his single one.

It was proved in Lemma 0:4.17 that for any variety $\mathcal{V}$ of regular semigroups, the class $\mathcal{V}^n = \{S: S^n \in \mathcal{V}\}$ also forms a semigroup variety. We point out that in general, $\mathcal{V}^n$ and $\mathcal{V}^{\inf(n)}$ are distinct classes, and $\mathcal{V}^{\inf(n)} \subseteq \mathcal{V}^{(n)}$.

The following conjecture, if proved correct, would be very useful in reducing the number of defining identities.

Conjecture 6:2.7 *Let* $\mathcal{V} = [x^{k+1} = x]$, *for some* $k \geq 1$.
Then for any $n \geq 1$,

$$\mathcal{V}^{\inf(n)} = [\, x_1x_2 \ldots x_n = (x_1)^{k+1}(x_2)^{k+1} \ldots (x_n)^{k+1} \,] \quad ❑$$

6:3 LATTICES OF INFLATED REGULAR SEMIGROUP VARIETIES

In this section, we show that for any variety $\mathcal{V}$ of n-inflated regular semigroups, the lattice $\mathcal{L}(\mathcal{V})$ of all subvarieties of $\mathcal{V}$ is isomorphic to the direct product $\mathcal{L}(\mathcal{V} \cap \mathcal{R}) \times \mathcal{L}(\mathcal{V} \cap \mathcal{N}_n)$, where $\mathcal{R}$ is the class of all regular semigroups, and $\mathcal{N}_n$ is the variety of all n-nilpotent semigroups.

For any variety $\mathcal{V} \subseteq \mathcal{W} \vee \mathcal{N}_n$, where $\mathcal{W}$ is a variety of regular semigroups, the class $\mathcal{V} \cap \mathcal{N}_n$ is clearly a variety. Concerning $\mathcal{V} \cap \mathcal{R}$, since both $\mathcal{V}$ and $\mathcal{R}$ are closed under taking arbitrary direct products and homomorphic images, so is the class $\mathcal{V} \cap \mathcal{R}$. We need only show that $\mathcal{V} \cap \mathcal{R}$ is also closed under taking subsemigroups. Accordingly, take any semigroup S in $\mathcal{V} \cap \mathcal{R}$. Then S is regular. Take any subsemigroup T of S. Since every member of $\mathcal{V}$ is an n-inflation of some regular semigroup in $\mathcal{W}$, by assumption, we have $\mathcal{V} \cap \mathcal{R} \subseteq \mathcal{W} \subseteq \mathcal{R}$. Since $\mathcal{W}$ is a variety, it is closed under subsemigroups and so T belongs to $\mathcal{W}$, and hence T is regular. Also, since $\mathcal{V}$ is closed under subsemigroups, T belongs in $\mathcal{V}$. We have thus shown that $T \in \mathcal{V} \cap \mathcal{R}$, proving the closure of $\mathcal{V} \cap \mathcal{R}$ under taking subsemigroups, and hence $\mathcal{V} \cap \mathcal{R}$ is a variety of semigroups.

Lemma 6:3.1 *Let $\mathcal{W}$ be a semigroup variety consisting entirely of regular semigroups. Then for any subvariety $\mathcal{V}$ of $\mathcal{W}^{\inf(n)}$*

$$\mathcal{V} = (\mathcal{V} \cap \mathcal{R}) \vee (\mathcal{V} \cap \mathcal{N}_n),$$

where $\mathcal{R}$ is the class of all regular semigroups.

Proof. Clearly, $(\mathcal{V} \cap \mathcal{R}) \vee (\mathcal{V} \cap \mathcal{N}_n) \subseteq \mathcal{V}$. We will prove only the converse. Take any semigroup S from $\mathcal{V}$. Let δ and $\ker\phi$ be the congruences as defined in the proof of Theorem 6:2.6. We have, by that same proof, that S is a subdirect product of S/δ (a member of $\mathcal{N}_n$) and $S/\mathrm{Ker}\phi$ (a regular semigroup in $\mathcal{W}$). But since $\mathcal{V}$ is a variety, it is closed under homomorphic images and so it follows that $S/\mathrm{Ker}\phi \in (\mathcal{V} \cap \mathcal{R})$ while $S/\delta \in (\mathcal{N}_n \cap \mathcal{V})$. We have thus proved the reverse containment, and the equality holds. ❑

Theorem 6:3.2 *For any fixed positive integer* n, *and any semigroup variety $\mathcal{W}$ consisting entirely of regular semigroups, the following map is a lattice isomorphism:*

$$\Phi_n : \mathcal{L}(\mathcal{W}) \quad \rightarrow \quad \mathcal{L}(\mathcal{W}^{\inf(n)}), \qquad \mathcal{V} \mapsto \mathcal{V}^{\inf(n)}.$$

Hence the intervals $[\mathcal{T},\mathcal{W}]$ and $[\mathcal{N}_n,\mathcal{W}^{\inf(n)}]$ are isomorphic.

Proof. Let $\mathcal{R}^{\inf(n)}$ denote the class of all n-inflated regular semigroups. For any subvarieties $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{W} \subseteq \mathcal{R}$,

$$\begin{aligned}
(\mathcal{V}_1 \cap \mathcal{V}_2)\Phi_n &= \{S \in \mathcal{R}^{\inf(n)} : S^n \in (\mathcal{V}_1 \cap \mathcal{V}_2)\} \\
&= \{S \in \mathcal{R}^{\inf(n)} : S^n \in \mathcal{V}_1 \text{ and } S^n \in \mathcal{V}_2\} \\
&= \{S \in \mathcal{R}^{\inf(n)} : S^n \in \mathcal{V}_1\} \cap \{S \in \mathcal{R}^{\inf(n)} : S^n \in \mathcal{V}_2\} \\
&= (\mathcal{V}_1)\Phi_n \cap (\mathcal{V}_2)\Phi_n;
\end{aligned}$$

and so Φ_n preserves varietal intersections. By making use of Theorem 6:2.6 for the definition of Φ_n, we have the following equalities, which show that Φ_n also preserves varietal joins

$$(\mathcal{V}_1 \vee \mathcal{V}_2)\Phi_n = (\mathcal{V}_1 \vee \mathcal{V}_2) \vee \mathcal{N}_n = (\mathcal{V}_1 \vee \mathcal{N}_n) \vee (\mathcal{V}_2 \vee \mathcal{N}_n) = (\mathcal{V}_1)\Phi_n \vee (\mathcal{V}_2)\Phi_n.$$

We will next show that Φ_n is also one-to-one. Now take any subvarieties $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{W}$ such that $\mathcal{V}_1^{\inf(n)} = \mathcal{V}_2^{\inf(n)}$. Suppose for a contradiction that

$\mathcal{V}_1 \neq \mathcal{V}_2$. Then without loss of generality, there exists Γ in $\mathcal{V}_1$ but not in $\mathcal{V}_2$. Take any non-trivial n-nilpotent semigroup N, and consider the direct product $S = N \times \Gamma$. For each element $a = (n,\alpha)$ of S, let $a^* = (0,\alpha')$, where 0 is the zero of N and α' is any inverse of the regular element α in Γ. Clearly, $S^n = \{0\} \times \Gamma = \{(0,\alpha): \alpha \in \Gamma\}$. Now, since for all elements $u = (k,\delta)$ in $S^{n-1} = N^{n-1} \times \Gamma$ and any element a of S, ua belongs to S^n,

$$ua = (k,\delta)(n,\alpha) = (0,\delta\alpha) = (0,\delta\alpha\alpha'\alpha) = (k,\delta)\,(n,\alpha)\,(0,\alpha')\,(n,\alpha) = uaa^*a;$$

and similarly, $au = aa^*au$. Now, these equalities together imply that $\delta_{n-1} = \theta(n-1,0) \cap \theta(0,n-1)$ is a regular congruence; and by Theorem 6:1.2 the map $\phi: S \to S, (x,y) \mapsto (0,y)$ is a homomorphism and $\ker\phi = \delta_{n-1}$. Since the set $Reg(S) = \{(0,\alpha): \alpha \in \Gamma\}$ forms a subsemigroup of S, we have by Theorem 3:1.6 that S is an n-inflation of Reg(S) which is isomorphic to Γ. Now, S does not belong in $\mathcal{V}_2^{\inf(n)}$ since $\Gamma \notin \mathcal{V}_2$, but clearly $S \in \mathcal{V}_1^{\inf(n)}$. By this contradiction, we must have $\mathcal{V}_1 = \mathcal{V}_2$.

It remains to show that Φ_n is onto. To prove this, take any variety $\mathcal{V}$ in the interval $[\mathcal{N}_n, \mathcal{W}^{\inf(n)}]$. Then from Lemma 6:2.6, this means that

$$\mathcal{N}_n \subseteq \mathcal{V} \subseteq \mathcal{W}^{\inf(n)} = \mathcal{N}_n \vee \mathcal{W}.$$

Now,

$$\begin{aligned} \mathcal{V} &= (\mathcal{N}_n \cap \mathcal{V}) \vee (\mathcal{V} \cap \mathcal{R}) && \text{(by Lemma 6:3.1)} \\ &= \mathcal{N}_n \vee (\mathcal{V} \cap \mathcal{W}) && \text{(since } \mathcal{N}_n \subseteq \mathcal{V} \text{ and } \mathcal{V} \cap \mathcal{W} = \mathcal{V} \cap \mathcal{R}) \\ &= (\mathcal{V} \cap \mathcal{W})^{\inf(n)} && \text{(by Theorem 6:2.6)} \\ &= (\mathcal{V} \cap \mathcal{W})\Phi_n; \end{aligned}$$

and hence the map Φ_n is onto since $\mathcal{V} \cap \mathcal{W} \subseteq \mathcal{W}$. Thus we have shown that the lattice interval $[\mathcal{T}, \mathcal{W}]$ is isomorphic to $[\mathcal{N}_n, \mathcal{W}^{\inf(n)}]$ under Φ_n. The lower limit of the latter interval can be obtained by Theorem 6:2.6. ❑

CHAPTER 7

VARIETIES OF STRUCTURALLY TRIVIAL SEMIGROUPS

In the final chapter of this thesis, certain recursive varietal relations involving structurally trivial semigroup varieties are shown to exist. Using these relations, the skeletons of certain structurally trivial semigroup varieties are described. In particular, the skeleton of the lattice of all subvarieties of the variety formed by all n-nilpotent extensions of rectangular bands, is shown to look somewhat like an inverted pyramid.

7:1 NOTATIONS AND BASIC CONCEPTS

Recall that a semigroup is *structurally trivial* if it belongs to the class

(7:1.1) $$\mathcal{T}^{(\infty,\infty)} = \bigcup \{\mathcal{T}^{(n,m)} : n \geq 0 \text{ and } m \geq 0\},$$

where $\mathcal{T}$ denotes the class of all trivial semigroups. In this section we will establish some concepts and notations to be used throughout the chapter.

For any non negative integers n, m, p and q, the symbols: $\boldsymbol{u}(\boldsymbol{n})$, $\boldsymbol{v}(\boldsymbol{m})$, $\boldsymbol{w}(\boldsymbol{p})$, and $\boldsymbol{t}(\boldsymbol{q})$ shall denote words over an infinite alphabet, say X, of length n, m, p, and q, respectively; and a letter from that alphabet occurs only once in any one of these words. Whenever these words are used to sandwich an identity, say

$$P_j(x_1, x_2, \ldots, x_{k_j}) = Q_j(x_1, x_2, \ldots, x_{k_j}),$$

by writing

$$\boldsymbol{v}(\boldsymbol{m})P_j(x_1, x_2, \ldots, x_{k_j})\boldsymbol{w}(\boldsymbol{p}) = \boldsymbol{v}(\boldsymbol{m})Q_j(x_1, x_2, \ldots, x_{k_j})\boldsymbol{w}(\boldsymbol{p}),$$

then we will assume that there are no common letters between those forming these words and with those forming the identity being sandwiched. This method of obtaining new identities is precisely the same as the one we described just prior to Theorem 2:2.6. The following well known varieties will be used in this chapter:

(7:1.2) $\mathcal{T} = [x = y]$ *trivial semigroups*

(7:1.3) $\mathcal{N}_2 = [xy = wt]$ *null semigroups*

(7:1.4) $\mathcal{N}_k = [\boldsymbol{w}(\boldsymbol{k}) = \boldsymbol{t}(\boldsymbol{k})]$ *k-nilpotent semigroups, k ≥ 1*

(7:1.5) $\mathcal{Z}_l = [xy = x]$ *left zero bands*

(7:1.6) $\mathcal{Z}_r = [xy = y]$ *right zero bands*

(7:1.7) $\mathcal{Z}_l \vee \mathcal{Z}_r = [xyx = x]$ *rectangular bands*

and $(\mathcal{Z}_l \vee \mathcal{Z}_r)^k = \{ S\colon S^k \in (\mathcal{Z}_l \vee \mathcal{Z}_r) \}, k \geq 1$

$$= [\, \boldsymbol{u}(\boldsymbol{k})\ \boldsymbol{v}(\boldsymbol{k})\ \boldsymbol{u}(\boldsymbol{k}) = \boldsymbol{u}(\boldsymbol{k})]$$

The following varieties ((7:1.8) — (7:1.14)) are obtained by considering the class $\mathcal{V}^{(n,m)} = \{S\colon S/\theta(n,m) \in \mathcal{V}\}$, where $\mathcal{V}$ is a variety taken from those given above ((7:1.2) — (7:1.7)).

(7:1.8) $\mathcal{T}^{(n,m)} = [\boldsymbol{u}(\boldsymbol{n})\ y\ \boldsymbol{v}(\boldsymbol{m}) = \boldsymbol{u}(\boldsymbol{n})\ x\ \boldsymbol{v}(\boldsymbol{m})]$

(7:1.9) $\mathcal{N}_2^{(n,m)} = [\boldsymbol{u}(\boldsymbol{n})\ xy\ \boldsymbol{v}(\boldsymbol{m}) = \boldsymbol{u}(\boldsymbol{n})\ wt\ \boldsymbol{v}(\boldsymbol{m})]$

(7:1.10) $\mathcal{N}_k^{(n,m)} = [\boldsymbol{u}(\boldsymbol{n})\ \boldsymbol{w}(\boldsymbol{k})\ \boldsymbol{v}(\boldsymbol{m}) = \boldsymbol{u}(\boldsymbol{n})\ \boldsymbol{t}(\boldsymbol{k})\ \boldsymbol{v}(\boldsymbol{m})]$

(7:1.11) $\mathcal{Z}_l^{(n,m)} = [\boldsymbol{u}(\boldsymbol{n})\ xy\ \boldsymbol{v}(\boldsymbol{m}) = \boldsymbol{u}(\boldsymbol{n})\ x\ \boldsymbol{v}(\boldsymbol{m})]$

(7:1.12) $\mathcal{Z}_r^{(n,m)} = [\boldsymbol{u}(\boldsymbol{n})\ xy\ \boldsymbol{v}(\boldsymbol{m}) = \boldsymbol{u}(\boldsymbol{n})\ y\ \boldsymbol{v}(\boldsymbol{m})]$

(7:1.13) $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(n,m)} = [\boldsymbol{u}(\boldsymbol{n})\ xyx\ \boldsymbol{u}(\boldsymbol{m}) = \boldsymbol{u}(\boldsymbol{n})\ x\ \boldsymbol{v}(\boldsymbol{m})]$

(7:1.14) $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n,m)}$

$$= [\boldsymbol{w}(\boldsymbol{n})\boldsymbol{u}(\boldsymbol{k})\ \boldsymbol{v}(\boldsymbol{k})\ \boldsymbol{u}(\boldsymbol{k})\boldsymbol{t}(\boldsymbol{m}) = \boldsymbol{w}(\boldsymbol{n})\ \boldsymbol{u}(\boldsymbol{k})\ \boldsymbol{t}(\boldsymbol{m})],\ k \geq 1$$

Such semigroups have appeared in the literature. For example, on Page 360 of Almeida (1986a), the varieties $\mathcal{Z}_l^{(k-1,0)}$, $\mathcal{Z}_r^{(k-1,0)}$, $\mathcal{Z}_l^{(k-1,k)}$ and $\mathcal{T}^{(\infty,\infty)}$ are each denoted there, respectively, by $\mathcal{K}_k$, $\mathcal{K}_k^r$, $\mathcal{N}erb_k$ and $\mathcal{N}erb$, in the context of pseudovarieties and generalised varieties. We observe from the identities that determine $\mathcal{Z}_l^{(k-1,k)}$ and $\mathcal{Z}_r^{(k,k-1)}$, that these varieties are equal, for every $k \geq 1$. We point out also, that the varieties given above are each determined by a single identity; and we observe that these semigroups are locally trivial in the sense that $eSe = \{e\}$ for every idempotent element e in S.

Theorem 7:1.15 (Almeida (1988(a)) *Let S be a semigroup:*

(i) If $S \in \mathcal{N}erb_k$ *then* S^{2k} *is a rectangular band.*

(ii) If S is a semigroup such that S^n *is a rectangular band, then* $S \in \mathcal{N}erb_{2n}$.

In particular, $S \in \mathcal{N}erb$ *if and only if* S *is a nilpotent extension of a rectangular band.* ❑

As before, we say S is a *nilpotent extension* of a rectangular band if S^n is a rectangular band for some $n \geq 1$. Since rectangular bands are structurally trivial, all of the following classes are what Almeida (1988(a)) terms $\mathcal{N}erb$:

$$\mathcal{T}^{(\infty,\infty)} = \mathcal{Z}_l^{(\infty,\infty)} = \mathcal{Z}_r^{(\infty,\infty)} = (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(\infty,\infty)}.$$

More generally, for any variety $\mathcal{V} \subseteq \mathcal{T}^{(\infty,\infty)}$, we have $\mathcal{V}^{(\infty,\infty)} = \mathcal{T}^{(\infty,\infty)}$.

The next result follows immediately from the above observations and Theorem 7:1.15.

Corollary 7:1.16 *A semigroup S is structurally trivial if and only if it is a nilpotent extension of some rectangular band.* ❑

The class $\mathcal{N}$ of all nilpotent semigroups is an important subclass of the class of all structurally trivial semigroups. These are semigroups for which $S^n = \{0\}$ for some positive integer n. Korjakov (1982) presented a sketch of the lattice of all varieties of commutative nilpotent semigroups. The author has not, as yet, seen a complete description of the lattice of all nilpotent semigroup varieties. However, Almeida and Reilly (1984) described completely a subvariety of this variety, namely the variety given in (0:4.14) defined by:

$$\mathcal{U}_n = [xy = yx, x^2 = y^2] \cap \mathcal{N}_n, \qquad \text{for every } n \geq 1.$$

They proved that the lattice of all subvarieties of $\mathcal{U}_n$ forms an ascending chain of length n-1.

We will denote by $\wp$ the family of all structurally trivial varieties, consisting of the following types:

(7:1.17)	$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(i,j)}$,	for $n \geq 1, i \geq 0, j \geq 0$
(7:1.18)	$\mathcal{Z}_l^{(i,j)}$,	for $i \geq 0, j \geq 0$
(7:1.19)	$\mathcal{Z}_r^{(i,j)}$,	for $i \geq 0, j \geq 0$
(7:1.20)	$\mathcal{N}_k^{(i,j)}$,	for $k \geq 1, i \geq 0, j \geq 0$.

A set $\mathcal{X}$ of varieties is said to form a *diamond* if $|\mathcal{X}| = 4$ such that two of its members are incomparable, while the other two form the greatest and the least members. If a diamond is closed under both varietal joins and meets, then we call it a *closed diamond*. We point out however, that a closed diamond may not be *convex* in the sense that there may be varieties occurring between two members of $\mathcal{X}$ which are not in $\mathcal{X}$. Examples of closed diamonds are given in Corollary 7:2.8, and in Figures 0:4.15 and 0:4.13.

For any variety $\mathcal{V}$ of structurally trivial semigroups in $\wp$, by the *skeleton* of the lattice $\mathcal{L}(\mathcal{V})$, we mean the lattice structure of the partially ordered set

$$\mathcal{L}_S(\mathcal{V}) = \{\mathcal{X} \in \wp : \mathcal{X} \subseteq \mathcal{V}\}.$$

In fact, $\mathcal{L}_S(\mathcal{V}) = \mathcal{L}(\mathcal{V}) \cap \wp$. It is not known whether or not the skeleton $\mathcal{L}_S(\mathcal{V})$ does not form a sublattice of the lattice $\mathcal{L}(\mathcal{V})$ in general. We do know, however, that $|\mathcal{L}_S(\mathcal{V})| \leq |\mathcal{L}(\mathcal{V})|$. Moreover, $\mathcal{L}_S(\mathcal{V})$ is a very crude but useful estimation of the lattice structure of $\mathcal{L}(\mathcal{V})$. Consider the variety $\mathcal{N}_{k+1}$ of all (k+1)-nilpotent semigroups. The skeleton $\mathcal{L}_S(\mathcal{N}_{k+1})$ forms a chain of length k, whereas, the lattice $\mathcal{L}(\mathcal{N}_{k+1})$ is not a chain. In fact, only the endpoints of the interval $[\mathcal{N}_k, \mathcal{N}_{k+1}]$ are included in $\wp$, and we know that there are many other varieties which do exist between them. We have defined the concept of skeleton only for varieties in $\wp$. And it is our purpose in this chapter to describe the lattice structure of (the partially ordered set) $\wp$. As we will see later, this is the first natural step to studying lattice structures of structurally trivial varieties. Structurally trivial varieties form an enormous class of semigroup varieties about which very little is known. Attempts have been made to study its subclasses such as the class of nilpotent semigroup varieties, but even for this, very little is known, and its lattice structure is thought to be very complicated. Our skeleton approach gives a nice framework on which we can add more structures, as information becomes available in the future. An extra bonus for our approach is that the skeleton $\mathcal{L}_S((\mathcal{Z}_1 \vee \mathcal{Z}_r)^n)$ is shown to be a disjoint union of what we call *planes*, and each one of these planes is closed under taking varietal joins and meets. Hence, the skeleton of $\mathcal{L}((\mathcal{Z}_1 \vee \mathcal{Z}_r)^n)$ is a disjoint union of a family of sublattices of the lattice of all semigroup varieties; and thus the concept of skeleton is not an artificial one.

For any variety $\mathcal{V}$ of semigroups, let $\mathcal{V}^k = \{S: S^n \in \mathcal{V}\}$, for $k \geq 1$. It was proved in Lemma 0:4.17 that $\mathcal{V}^k$ is also a variety whenever $\mathcal{V}$ is. Of course, $\mathcal{V}^k$ consists precisely of all k-nilpotent extensions of semigroups in $\mathcal{V}$. In Sections 7:2, 7:3 and 7:4, certain recursive relations are shown to exist between members of $\wp$. In Section 7:5, the skeleton of $\mathcal{L}((\mathcal{Z}_1 \vee \mathcal{Z}_r)^3)$ is constructed. Then in Section 7:6, the skeleton of $\mathcal{L}((\mathcal{Z}_1 \vee \mathcal{Z}_r)^n)$ is modelled. This skeleton looks somewhat, superficially, like an inverted pyramid. In general, describing lattices of semigroup varieties are not easy problems, and our description of the skeleton of $\mathcal{L}((\mathcal{Z}_1 \vee \mathcal{Z}_r)^n)$ successfully records certain observations. We hope that with further research in the future, a more complete picture of this lattice will emerge.

7:2 RECURSIVE RELATIONS INVOLVING $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j)}$

In this section, we plan to describe the interrelationships that exist among varieties of the type (7:1.17) in $\wp$. Note that semigroups in the variety $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j)}$ consists of all those for which $S/\theta(i,j)$ belong to the variety $(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n$.

Lemma 7:2.1 *The following containment relations hold for every $n \geq 2$:*

(a) $\mathcal{T}^{(n,0)} \subseteq \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)}$

(b) $\mathcal{Z}_1^{(n-1,0)} \subseteq (\mathcal{Z}_1 \vee \mathcal{Z}_r)^n$.

(c) $\mathcal{Z}_1^{(n-1,0)} \subset \mathcal{T}^{(n,0)}$

Proof. (a). Take any elements: s,t, $u = u_1u_2u_3 \ldots u_{n-1}$ and $d = d_1d_2d_3 \ldots d_{n-1}$ of a semigroup S in $\mathcal{T}^{(n,0)}$. Then

$$sut = (su_1u_2u_3 \ldots u_{n-1})\, t = (su_1u_2u_3 \ldots u_{n-1})(du)t = s(udu)t;$$

and so $S \in \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)}$, proving that (a) holds.

(b) Take any $S \in \mathcal{Z}_l{}^{(n-1,0)}$ and any elements $s = s_1s_2 \ldots s_n$ and $t = t_1t_2 \ldots t_n$ of S.

Then

$$s = (s_1s_2 \ldots s_{n-1})s_n = (s_1s_2 \ldots s_{n-1})s_n(ts) = sts;$$

and so $S \in (\mathcal{Z}_l \vee \mathcal{Z}_r)^n$, proving that (b) also holds.

Since every left zero band satisfies the identity $zx = zy$, the containment $\mathcal{Z}_l \subseteq \mathcal{T}^{(1,0)}$ holds. Since $\mathcal{T}^{(1,0)}$ is a larger variety which includes all null semigroups, the containment $\mathcal{Z}_l \subset \mathcal{T}^{(1,0)}$ is strict. The conclusion (c) is in fact the image of the strict containment $\mathcal{Z}_l \subset \mathcal{T}^{(1,0)}$ under the strict containment preserving lattice map $\mathcal{V} \mapsto \mathcal{V}^{(n-1,0)}$.

Lemma 7:2.2 *The following (strict) varietal containments hold for every ordered pair (i,j) of non-negative integers:*

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(i,j)} \subset \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(i+1,j+1)}, \qquad \textit{for } n = 2,3,4,5,\ldots$$

Proof. The varieties $(\mathcal{Z}_l \vee \mathcal{Z}_r)^n$ and $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)}$ are each determined, respectively, by the identities:

$$\mathbf{u(n)}\ \mathbf{v(n)}\ \mathbf{u(n)} = \mathbf{u(n)} \text{ and } \quad z\,\mathbf{u(n-1)}\ \mathbf{v(n-1)}\ \mathbf{u(n-1)}\ k = z\,\mathbf{u(n-1)}\ k.$$

Take any semigroup S from $(\mathcal{Z}_l \vee \mathcal{Z}_r)^n$ and any elements $a = a_1a_2a_3 \ldots a_{n-1}$, $b = b_1b_2b_3 \ldots b_{n-1}$, s and t of S. Then since S^n is a rectangular band,

$$\begin{aligned} sat = (sa)t &= [\,(sa)\,(batb)\,(sa)\,]\,t \qquad (\text{since } sa, batb \in S^n) \\ &= (sab)\,[\,(at)\,(bs)\,(at)\,] \\ &= (sab)\,(at) \qquad\qquad (\text{since } at, bs \in S^n) \\ &= sabat; \end{aligned}$$

and hence $(\mathcal{Z}_l \vee \mathcal{Z}_r)^n \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)}$. We will prove the strictness of this containment. Accordingly, from Lemma 7:2.1(a) and Lemma 7:2.1(b), the following containments hold:

$$\mathcal{T}^{(n,0)} \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)} \qquad \text{and} \qquad \mathcal{Z}_l^{(n-1,0)} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n};$$

and that strict containment $\mathcal{Z}_l^{(n-1,0)} \subset \mathcal{T}^{(n,0)}$ holds by Lemma 7:2.1(c). Now, the varieties $\mathcal{Z}_l^{(n-1,0)}$ and $\mathcal{T}^{(n,0)}$ are each determined, respectively, by the identities

$$\mathbf{u}(\mathbf{n-1})xy = \mathbf{u}(\mathbf{n-1})x \quad \text{and} \quad \mathbf{u}(\mathbf{n})x = \mathbf{u}(\mathbf{n})y.$$

By the strictness of the containment $\mathcal{Z}_l^{(n-1,0)} \subset \mathcal{T}^{(n,0)}$, there exists a semigroup S from $\mathcal{T}^{(n,0)}$ and elements $b \in S$ and $s, t \in S^n$ such that $s \neq sb = t$. For a contradiction, suppose that $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)}$ is contained in $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}$. It then follows that S^n is a rectangular band, and the elements s and t are both idempotents. By the idempotency of t,

$$s = s(bsb)s = (sb)(sb)s = tts = ts \text{ and } \quad st = s(sb) = sb = t,$$

and so {s,t} forms a right zero band. Since S satisfies the identity

$$\mathbf{u}(\mathbf{n})x = \mathbf{v}(\mathbf{n})y,$$

we have $t = t^2 = tt = ts = s$, which contradicts the fact that the elements s and t are distinct. We have thus proved the strictness of the containment

$$(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n} \subset \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)}.$$

The conclusion follows by taking the image of this containment under the lattice map $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$ (which preserves strict containments). ❑

Lemma 7:2.3 *For every $n \geq 1$, the following recursive varietal relations (equalities) exist:*

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}\}^{(1,0)} \vee \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}\}^{(0,1)} = (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1}.$$

Proof. Let $\mathcal{X} = \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}\}^{(1,0)} \vee \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}\}^{(0,1)}$ and $\mathcal{Y} = (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1}$. We will show that $\mathcal{X} \subseteq \mathcal{Y}$. Take any semigroup $S \in \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}\}^{(1,0)}$, and any elements

$a = a_1a_2a_3 \ldots a_{n+1}$, $\quad b = b_1b_2b_3 \ldots b_{n+1}$ $\quad$ and $\quad c = a_2a_3 \ldots a_{n+1}$ of S.

Since $a = a_1c$, and S satisfies the identity

$$z\,\mathbf{u}(\mathbf{n})\,\mathbf{v}(\mathbf{n})\,\mathbf{u}(\mathbf{n}) = z\,\mathbf{u}(\mathbf{n}),$$

we have

$$a = a_1c = a_1c(ba_1)\,c = (a_1c)\,b\,(a_1c) = aba.$$

Hence, $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}\}^{(1,0)} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1}$ Dually, $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}\}^{(0,1)}$ is also contained in $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1}$. And these containments, together, imply that the join of

$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(1,0)}$ and $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(0,1)}$ is contained in $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1}$. Hence $\mathcal{X} \subseteq \mathcal{Y}$. Conversely, take any $S \in (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1}$, and consider the congruence:

$$\delta_1 = \theta^S(1,0) \cap \theta^S(0,1) = \{(a,b) : xa = xb,\ ay = by \text{ for all } x,y \in S\}.$$

Then δ_1 is idempotent-separating, since for all (e,f) in $\delta_1 \cap (E(S) \times E(S))$,

$$e = ee = ef = ff = f.$$

The map $a\delta_1 \mapsto (a\theta(1,0), a\theta(0,1))$, with $a \in S$, is an embedding of S/δ_1 into the direct product $S/\theta(1,0) \times S/\theta(0,1)$. Now, we observe that by putting $i = 0 = j$ in Lemma 7:2.2, we have $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1} \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(1,1)}$, and so

$$S/\theta(1,0) \in \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(0,1)} \qquad \text{and} \qquad S/\theta(0,1) \in \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(1,0)}.$$

It follows that

$$S/\delta_1 \in \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(1,0)} \vee \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(0,1)}.$$

Now the varieties $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(1,0)}$ and $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(0,1)}$ are each determined respectively, by the single identities,

$$z\mathbf{u}(\mathbf{n})\ \mathbf{v}(\mathbf{n})\ \mathbf{u}(\mathbf{n}) = z\mathbf{u}(\mathbf{n}) \qquad \text{and} \qquad \mathbf{u}(\mathbf{n})\ \mathbf{v}(\mathbf{n})\ \mathbf{u}(\mathbf{n})z = \mathbf{u}(\mathbf{n})z.$$

These identities are formed by words of length n+1 and 3n+1, and so in view of Lemma 0:4.21, every word forming an identity which defines

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(1,0)} \vee \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(0,1)}$$

will be of length at least n+1. Let

$$(\dagger) \qquad P(x_1,x_2,x_3,\ldots,x_k) = Q(x_1,x_2,x_3,\ldots,x_k),$$

with $k \geq n+1$, be such an identity. Then for any choice of elements $s_1,s_2,s_3,\ldots,s_k$ in S, we know that

$$P(s_1\delta_1,s_2\delta_1,s_3\delta_1,\ldots,s_k\delta_1) = Q(s_1\delta_1,s_2\delta_1,s_3\delta_1,\ldots,s_k\delta_1).$$

Thus $(P(s_1,s_2,s_3,\ldots,s_k),\ Q(s_1,s_2,s_3,\ldots,s_k)) \in \delta_1$. However, since $S \in (\mathcal{Z}_l \vee \mathcal{Z}_r)^n$ both the elements $P(s_1,s_2,s_3,\ldots,s_k)$ and $Q(s_1,s_2,s_3,\ldots,s_k)$ are idempotents in S, and we observed that δ_1 is idempotent-separating.
Thus

$$P(s_1,s_2,s_3,\ldots,s_k) = Q(s_1,s_2,s_3,\ldots,s_k),$$

and so the identical relation $(\dagger)$ holds in S. It follows that

$$S \in \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(1,0)} \vee \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(0,1)},$$

as required. ❑

The method we have employed to prove the above lemma, in particular the part concerning the use of δ_1, will be adapted later to prove other results in similar circumstances. For this reason, we will hereafter refer to the idea of the proof as the *bisection method* or simply the *bisective* argument.

Lemma 7:2.4 *The following recursive varietal relation (equalities) exist:*

$$\{(Z_l \vee Z_r)^n\}^{(1,0)} \cap \{(Z_l \vee Z_r)^n\}^{(0,1)} = \begin{cases} Z_r^{(1,0)} & , \text{ if } n = 1 \\ \{(Z_l \vee Z_r)^{n-1}\}^{(1,1)} , & \text{if } n = 2,3,4,\dots. \end{cases}$$

Proof. Case $n \geq 2$: Take any semigroup $S \in \{(Z_l \vee Z_r)^n\}^{(1,0)} \cap \{(Z_l \vee Z_r)^n\}^{(0,1)}$, and consider the elements x, y, $s = s_1s_2s_3 \dots s_{n-1}$, and $t = t_1t_2t_3 \dots t_{n-1}$ of S.

Then

$$xsy = x(sy) = x[(sy)(txst)(sy)] = [(xs)(yt)(xs)]\,(tsy) = (xs)(tsy) = x(sts)\,y.$$

This proves that S satisfies the identity

$$z\,\mathbf{u}(n-1)\,\mathbf{v}(n-1)\,\mathbf{u}(n-1)\,k = z\,\mathbf{u}(n-1)\,k,$$

and hence $S \in \{(Z_l \vee Z_r)^{n-1}\}^{(1,1)}$. Conversely, take any $S \in \{(Z_l \vee Z_r)^{n-1}\}^{(1,1)}$, and any elements $a = a_1a_2 \dots a_n$, $b = b_1b_2 \dots b_n$, $c = a_2a_3 \dots a_n$ and z in S.
Then

$$az = a_1 c z = a_1 [c\,(ba_1)\,c]\,z = (a_1c)\,b\,(a_1 c)z = abaz;$$

and hence $S \in \{(Z_l \vee Z_r)^n\}^{(0,1)}$. We have thus shown that $\{(Z_l \vee Z_r)^{n-1}\}^{(1,1)} \subseteq \{(Z_l \vee Z_r)^n\}^{(0,1)}$. Dually, $\{(Z_l \vee Z_r)^{n-1}\}^{(1,1)} \subseteq \{(Z_l \vee Z_r)^n\}^{(1,0)}$. By combining these containments ,

$$\{(Z_l \vee Z_r)^{n-1}\}^{(1,1)} \subseteq (Z_l \vee Z_r)^n\}^{(1,0)} \cap (Z_l \vee Z_r)^n\}^{(0,1)},$$

and that completes the proof for the case $n \geq 2$.

Consider now the case n =1. Take any $S \in (Z_l \vee Z_r)\}^{(1,0)} \cap (Z_l \vee Z_r)\}^{(0,1)}$. Then both $S/\theta(1,0)$ and $S/\theta(0,1)$ are rectangular bands. Since rectangular bands satisfy the identities $x^2 = x$ and $zxy = zy$, it follows by Theorem 2:2.6 that S satisfies $zx^2 = zx$, $x^2 z = xz$, $wzxy = wzy$ and $zxyk = zyk$. Now, for any elements a, b, c of S we have

$$abc = a(bc)^2 = abcbc = (abc)bc = (ac)bc = a(cbc) = ac.$$

Hence S satisfies the identity abc = ac, which proves that $S \in \mathcal{Z}_r^{(1,0)}$. Conversely, take any $S \in \mathcal{Z}_r^{(1,0)} = [zxy = zy]$. Then for any elements a, b, c of S, a(bcb) = a(bc)b = ab and (bcb)a = b(cb)a = ba. These equalities, together, prove that S belongs to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)\}^{(1,0)} \cap \{(\mathcal{Z}_l \vee \mathcal{Z}_r)\}^{(0,1)}$, and that completes the proof for the case n = 1. ❑

Lemma 7:2.5 *The following (strict) varietal containments hold for any ordered pair (i,j) of positive integers:*

(a) $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(i,j)} \subset (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+i+j}$, *for* $n = 1,2,3,\ldots$

(b) $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(i+1,j+1)} \subset \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(i+1,j)}$, *for* $n = 2,3,\ldots$

and

$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(i+1,j+1)} \subset \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(i,j+1)}$, *for* $n = 2,3,\ldots$

Proof. (a) Take any semigroup S from $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(i,j)}$, and consider the following elements of S:

$s = s_1s_2\ldots s_{n+i+j}$, $t = t_1t_2\ldots t_{n+i+j}$, $g = s_1s_2\ldots s_i$, $h = s_{i+n+1}s_{i+n+2}\ldots s_{i+n+j}$, and $c = (s_{i+n+1}s_{i+n+2}\ldots s_{i+n+j})\, t\, (s_1s_2s_2\ldots s_i) = htg$.

Then

$$\begin{aligned} s = s_1s_2\ldots s_{n+i+j} &= (s_1s_2\ldots s_i)(s_{i+1}s_{i+2}\ldots s_{n+i})(s_{i+n+1}s_{i+n+2}\ldots s_{i+n+j}) \\ &= g\,(s_{i+1}s_{i+2}\ldots s_{n+i})\,h \\ &= g\,[(s_{i+1}s_{i+2}\ldots s_{n+i})\,c\,(s_{i+1}s_{i+2}\ldots s_{n+i})]\,h \\ &= (g\,s_{i+1}s_{i+2}\ldots s_{n+i}h)\,t\,(gs_{i+1}s_{i+2}\ldots s_{n+i}h) \qquad \text{(since } c = htg) \\ &= s\,t\,s \qquad \text{(since } s = g\,s_{i+1}s_{i+2}\ldots s_{n+i}h); \end{aligned}$$

and so $S \in (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+i+j}$. Having proved the containment, we now proceed to show its strictness. First, we observe from Lemma 7:2.3 that the following two containments are strict:

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(1,0)} \subset (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1} \quad \text{and} \quad \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^n\}^{(0,1)} \subset (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n+1}$$

Now, the sequence of containments given below are consequences of the above two:

$$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j)} \subset \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1}\}^{(i-1,j)} \subset \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+2}\}^{(i-2,j)} \ldots\ldots$$
$$\subset \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+i}\}^{(0,j)} \subset \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+i+1}\}^{(0,j-1)} \ldots\ldots\ldots$$
$$\subset \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+i+j}\}^{(0,0)} = (\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+i+j}.$$

(b) From Lemma 7:2.4 we have the following strict containments:

$$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)} \subset \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(1,0)}, \text{ for } n = 2,3,\ldots$$

and

$$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n-1}\}^{(1,1)} \subset \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(0,1)}, \text{ for } n = 2,3,\ldots$$

We have the conclusion by taking their image under $\mathcal{V} \mapsto \mathcal{V}^{(i-1,j-1)}$. ❑

Theorem 7:2.6 *For any positive integers n, and any non negative integers i and j, the following equalities hold:*

$$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i+1,j)} \vee \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j+1)} = \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1}\}^{(i,j)}, \qquad \textit{for } n=1,2,3,\ldots$$

Proof. From Lemma 7:2.3, both $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(1,0)}$ and $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(0,1)}$ are contained in $(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1}$. By the containment preserving nature of the map $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$, both $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i+1,j)}$ and $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j+1)}$ are contained in $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1}\}^{(i,j)}$. Hence,

$$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i+1,j)} \vee \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j+1)} \subseteq \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1}\}^{(i,j)}, \; \textit{for } n = 1,2,3,\ldots$$

To prove the reverse containment, we observe that putting i = 0 = j and by replacing n-1 by n in Lemma 7:2.2 we have $(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1} \subseteq \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(1,1)}$. Then, by the containment preserving nature of the maps $\mathcal{V} \mapsto \mathcal{V}^{(i-1,j)}$ and $\mathcal{V} \mapsto \mathcal{V}^{(i,j-1)}$, we have

(¥) $$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1}\}^{(i-1,j)} \subseteq \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j+1)}$$

and

(§) $$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1}\}^{(i,j-1)} \subseteq \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i+1,j)}.$$

Now, take any $S \in \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n+1}\}^{(i,j)}$, and consider the congruence $\delta_1 = \theta(1,0) \cap \theta(0,1)$ as in Lemma 7:2.3. Then as in that proof, S/δ_1 can be embedded into the direct product $S/\theta(1,0) \times S/\theta(0,1)$ under the mapping $a\delta_1 \mapsto (\theta(1,0),\theta(0,1))$.

Now from (¥) ,

$S/\theta(1,0) \in \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j+1)}$; and $S/\theta(0,1) \in \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i+1,j)}$ from (§).

The varieties $\{(Z_1 \vee Z_r)^n\}^{(i,j+1)}$ and $\{(Z_1 \vee Z_r)^n\}^{(i+1,j)}$ are each determined, respectively, by the identities:

$$w(i)\ u(n)\ v(n)\ u(n)\ t(j+1) = w(i)\ u(n)\ t(j+1)$$

and

$$w(i+1)\ u(n)\ v(n)\ u(n)\ t(j) = w(i+1)\ u(n)\ t(j);$$

and both these are formed by words of length at least n+1+j+n and 3n+1+i+j. Thus in view of Lemma 0:4.21, the words forming any defining identity of the join

$$\{(Z_1 \vee Z_r)^n\}^{(i+1,j)} \vee \{(Z_1 \vee Z_r)^n\}^{(i,j+1)}$$

are of length at least i+j+1+n. It follows, as proved in Lemma 7:2.3, that $S/\delta_1 \in \{(Z_1 \vee Z_r)^n\}^{(i+1,j)} \vee \{(Z_1 \vee Z_r)^n\}^{(i,j+1)}$. One can show by the bisective argument that S satisfies every identity satisfied by S/δ_1 that is formed by words of length at least n+1+i+j. Thus it follows that

$$\{(Z_1 \vee Z_r)^n\}^{(i,j)} \subseteq \{(Z_1 \vee Z_r)^n\}^{(i+1,j)} \vee \{(Z_1 \vee Z_r)^n\}^{(i,j+1)};$$

and that completes the proof. ❑

Theorem 7:2.7 *For any ordered pair (i,j) of non negative integers, the following equalities hold:*

$$\{(Z_1 \vee Z_r)^n\}^{(i+1,j)} \cap \{(Z_1 \vee Z_r)^n\}^{(i,j+1)} = \begin{cases} Z_r^{(i+1,j)} & , \ \text{if } n = 1 \\ \{(Z_1 \vee Z_r)^{n-1}\}^{(i+1,j+1)} & , \ \text{if } n = 2,3,\ldots \end{cases}$$

Proof. We observed in Lemma 7:2.4 that the following equalities hold:

$$\{(Z_1 \vee Z_r)^n\}^{(1,0)} \cap \{(Z_1 \vee Z_r)^n\}^{(0,1)} = \begin{cases} Z_r^{(1,0)} & , \ \text{if } n = 1 \\ \{(Z_1 \vee Z_r)^{n-1}\}^{(1,1)} & , \ \text{if } n = 2,3,4,\ldots \end{cases}$$

The conclusion follows by taking the image of these equalities under the $\cap$-preserving map $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$. ❑

The following result will be useful later in Sections 7:5 and 7:6.

Corollary 7:2.8 *The skeleton of the interval*

$$[\{(Z_1 \vee Z_r)^{n-1}\}^{(i+1,j+1)}, \{(Z_1 \vee Z_r)^{n+1}\}^{(i,j)}]$$

forms the following closed diamonds, where (i,j) is an ordered pair of non negative integers:

(a) for the case $n \geq 2$

$$\begin{tikzcd}[column sep=small]
 & \{(Z_1 \vee Z_r)^{n+1}\}^{(i,j)} \arrow[dl, dash] \arrow[dr, dash] & \\
\{(Z_1 \vee Z_r)^{n}\}^{(i+1,j)} \arrow[dr, dash] & & \{(Z_1 \vee Z_r)^{n}\}^{(i,j+1)} \arrow[dl, dash] \\
 & \{(Z_1 \vee Z_r)^{n-1}\}^{(i+1,j+1)} &
\end{tikzcd}$$

(b) and , for the case $n = 1$,

$$\begin{tikzcd}[column sep=small]
 & \{(Z_1 \vee Z_r)^{2}\}^{(i,j)} \arrow[dl, dash] \arrow[dr, dash] & \\
(Z_1 \vee Z_r)^{(i+1,j)} \arrow[dr, dash] & & (Z_1 \vee Z_r)^{(i,j+1)} \arrow[dl, dash] \\
 & Z_r^{(i+1,j)} &
\end{tikzcd}$$

❑

7:3 RECURSIVE RELATIONS INVOLVING $\mathcal{N}_k^{(i,j)}$

Here we study the relationships that exist between varieties of the type $\mathcal{N}_k^{(i,j)}$ in $\wp$. Note that semigroups in the variety $\mathcal{N}_k^{(i,j)}$ are precisely those for which $S/\theta(i,j)$ is a k-nilpotent semigroup.

Lemma 7:3.1 *For any positive integer* k, *the following recursive varietal relation (equality) holds*

$$\mathcal{N}_k^{(1,0)} \cap \mathcal{N}_k^{(0,1)} = \mathcal{N}_{k+1}\,.$$

Proof. Take any $S \in \mathcal{N}_k^{(1,0)} \cap \mathcal{N}_k^{(0,1)}$, and consider $s = s_1 s_2 \ldots s_{k+1}$, $t = t_1 t_2 t_3 \ldots\ldots t_{k+1} \in S$.

Then

$$s = (s_1 s_2 s_3 \ldots\ldots s_k)\, s_{k+1} = (t_1 t_2 t_3 \ldots\ldots t_k)\, s_{k+1} = t_1 (t_2 t_3 \ldots\ldots t_k t_{k+1}) = t\,.$$

We have the second equality since $S \in \mathcal{N}_k^{(0,1)}$; and the third equality since $S \in \mathcal{N}_k^{(1,0)}$. We have thus shown that S is a (k+1)-nilpotent semigroup.

Conversely, take any (k+1)-nilpotent semigroup S. Then since $S^{k+1} = \{0\}$, we see that for all $x \in S^k$ and any $a,b \in S$,

$$xa = 0 = xb \qquad \text{and} \qquad ax = 0 = bx.$$

Hence both $S/\theta(k,0)$ and $S/\theta(0,k)$ are trivial. We have thus proved that $S \in \mathcal{N}_k^{(1,0)} \cap \mathcal{N}_k^{(0,1)}$, and that completes the proof. ❑

Lemma 7:3.2 *The following recursive varietal relations (equalities) hold:*

$$\mathcal{N}_k^{(1,0)} \vee \mathcal{N}_k^{(0,1)} = \begin{cases} \mathcal{Z}_r^{(1,0)} & , \quad \text{if } k = 1 \\ \mathcal{N}_{k-1}^{(1,1)} & , \quad \text{if } k = 2,3,4\ldots \end{cases}$$

Proof. **Case k = 1:** Since $\mathcal{N}_1 = \mathcal{T}$, we must prove the equality

$$\mathcal{T}^{(1,0)} \vee \mathcal{T}^{(0,1)} = \mathcal{Z}_r^{(1,0)}\,.$$

We observe that

$$\mathcal{T}^{(1,0)} = [zx = zy] \qquad \text{and} \qquad \mathcal{T}^{(1,0)} = [xz = yz].$$

It is well known (see Figure 0:4.15), that the join of these equational classes is the variety $\mathcal{Z}_r^{(1,0)} = [zxy = zy]$, and that takes care of the case where k = 1.

Case $k \geq 2$: Take any $S \in \mathcal{N}_{k-1}^{(1,0)}$ and any $a = a_1a_2 \ldots a_{k-1}$, $b = b_1b_2 \ldots b_{k-1}$, x and y in S. Then

$$xay = x(a_1a_2 \ldots a_{k-1}\, y) = x(b_1b_2 \ldots b_{k-1}y) = xby;$$

and so $\mathcal{N}_k^{(1,0)} \subseteq \mathcal{N}_{k-1}^{(1,1)}$. Dually, $\mathcal{N}_k^{(0,1)} \subseteq \mathcal{N}_{k-1}^{(1,1)}$. Hence, $\mathcal{N}_{k-1}^{(1,1)}$ contains the join $\mathcal{N}_k^{(1,0)} \vee \mathcal{N}_k^{(0,1)}$. To prove the reverse inclusion, take any semigroup S in $\mathcal{N}_{k-1}^{(1,1)}$. Then

$$S/\theta(1,0) \in \mathcal{N}_{k-1}^{(0,1)} \subseteq \mathcal{N}_k^{(0,1)} \text{ and } S/\theta(0,1) \in \mathcal{N}_{k-1}^{(1,0)} \subseteq \mathcal{N}_k^{(1,0)}.$$

Let $\delta_1 = \theta(1,0) \cap \theta(0,1)$. Then, proceeding in the same way as in the proof of Lemma 7:2.3 and Theorem 7:2.6, one can show, using the bisective argument, that S satisfies every identity that determines the join $\mathcal{N}_k^{(1,0)} \vee \mathcal{N}_k^{(0,1)}$, and that completes the proof. ❑

Theorem 7:3.3 *The following equalities hold for any ordered pair (i,j) of non negative integers :*

$$\mathcal{N}_k^{(i+1,j)} \vee \mathcal{N}_k^{(i,j+1)} = \begin{cases} \mathcal{Z}_r^{(i+1,j)} , & \text{if } k = 1 \\ \mathcal{N}_{k-1}^{(i+1,j+1)} , & \text{if } k = 2, 3, 4, \ldots \end{cases}$$

Proof. **Case $k \geq 2$:** For any semigroup $S \in \mathcal{N}_{k-1}^{(i+1,j+1)}$, we observe that

$$S/\theta(1,0) \in \mathcal{N}_{k-1}^{(i,j+1)} \subseteq \mathcal{N}_k^{(i,j+1)} \quad \text{and} \quad S/\theta(0,1) \in \mathcal{N}_{k-1}^{(i+1,j)} \subseteq \mathcal{N}_k^{(i+1,j)}.$$

Let $\delta_1 = \theta(1,0) \cap \theta(0,1)$. As in the proof of Lemma 7:2.3, S/δ_1 can be embedded into the direct product $S/\theta(1,0) \times S/\theta(0,1)$, which, in turn, belongs to the variety $\mathcal{N}_k^{(i,j+1)} \vee \mathcal{N}_k^{(i+1,j)}$. The defining identities for $\mathcal{N}_k^{(i,j+1)}$ and $\mathcal{N}_k^{(i+1,j)}$ are formed by words of length at least k+i+j+1, and so in view of Lemma 0:4.21, the join $\mathcal{N}_k^{(i,j+1)} \vee \mathcal{N}_k^{(i+1,j)}$ must also have this property. It can shown, using the bisective argument, that S satisfies every identity formed by words of length at least k+i+j+1 that is satisfied by S/δ_1. Hence, it follows that S belongs to $\mathcal{N}_k^{(i,j+1)} \vee \mathcal{N}_k^{(i+1,j)}$.

Conversely, since $\mathcal{N}_k \subseteq \mathcal{N}_{k-1}^{(1,0)}$, and the map $\mathcal{V} \mapsto \mathcal{V}^{(i,j+1)}$ preserves varietal containments, $\mathcal{N}_{k-1}^{(i,j+1)} \subseteq \mathcal{N}_{k-1}^{(i+1,j+1)}$. Dually, $\mathcal{N}_{k-1}^{(i+1,j)} \subseteq \mathcal{N}_{k-1}^{(i+1,j+1)}$. These containments together prove that $\mathcal{N}_k^{(i,j+1)} \vee \mathcal{N}_k^{(i+1,j)} \subseteq \mathcal{N}_{k-1}^{(i+1,j+1)}$; and that proves the case $k \geq 2$.

Case k=1: Recall that $\mathcal{N}_1 = \mathcal{T}$. Since $\mathcal{T} \subset \mathcal{Z}_r$ and $\mathcal{T} \subset \mathcal{Z}_l$ it follows by the containment preserving nature of the maps $\mathcal{V} \mapsto \mathcal{V}^{(i,j+1)}$ and $\mathcal{V} \mapsto \mathcal{V}^{(i+1,j)}$ that both $\mathcal{T}^{(i+1,j)}$ and $\mathcal{T}^{(i,j+1)}$ are contained in $Z_r^{(i+1,j)} = Z_l^{(i,j+1)}$. Hence, $\mathcal{N}_1^{(i,j+1)} \vee \mathcal{N}_1^{(i+1,j)} \subseteq Z_r^{(i+1,j+1)}$. The reverse containment can be proved using bisective arguments.

❑

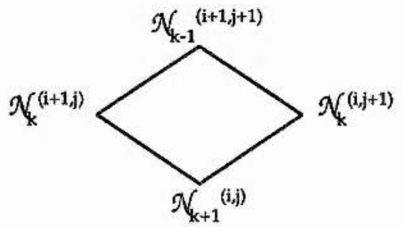
Theorem 7:3.4 *The following equality holds for any ordered pair (i,j) of non negative integers and* $k \geq 1$:

$$\mathcal{N}_k^{(i,j+1)} \bigcap \mathcal{N}_k^{(i+1,j)} = \mathcal{N}_{k+1}^{(i,j)}.$$

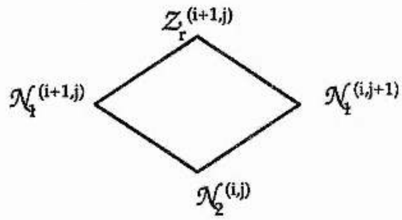
Proof. From Lemma 7:3.1, $\mathcal{N}_k^{(1,0)} \bigcap \mathcal{N}_k^{(0,1)} = \mathcal{N}_{k+1}$. The conclusion follows by the $\bigcap$-preserving nature of the map $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$. ❑

Corollary 7:3.5 *The skeleton of the lattice interval* $[\mathcal{N}_k^{(i,j)}, \mathcal{N}_{k-1}^{(i+1,j+1)}]$, *forms a closed diamond. In fact we have the following cases:*

(a) For the case $k = 2, 3, \ldots$



(b) and, for the case $k = 1$, *we have*



❑

We finally remark that since $\mathcal{N}_1 = \mathcal{T}$, the recursive relations involving varieties of the type $\mathcal{T}^{(i,j)}$ were taken care of as special cases of what we have already proved, and so there is nothing further to prove.

7:4 RECURSIVE RELATIONS INVOLVING $\mathcal{Z}_l^{(i,j)}$ AND $\mathcal{Z}_r^{(i,j)}$

We begin with the following useful relation.

Lemma 7:4.1 *Except for the cases:* $\mathcal{Z}_l^{(1,0)}$, $\mathcal{Z}_r^{(0,1)}$, $\mathcal{Z}_l$ *and* $\mathcal{Z}_r$, *the following equalities holds :*

$$\mathcal{Z}_r^{(i,j)} = \mathcal{Z}_l^{(i-1,j+1)} \text{ and } \mathcal{Z}_l^{(i,j)} = \mathcal{Z}_r^{(i+1,j-1)} .$$

Proof. From (7:1.11) and (7:1.12), $\mathcal{Z}_r^{(i,j)}$ and $\mathcal{Z}_l^{(i-1,j+1)}$ are each determined respectively by the identities

$$\mathbf{u}(i)xy\mathbf{v}(j) = \mathbf{u}(i)y\mathbf{v}(j) \quad \text{and} \quad \mathbf{u}(i-1)xy\mathbf{v}(j+1) = \mathbf{u}(i-1)x\mathbf{v}(j+1).$$

Clearly, by the associativity, these identities are equal, though expressed differently, and so the varieties they represent are also equal. This proves the first equality. The second equality follows by symmetry. ❑

Theorem 7:4.2 *For any ordered pair (i,j) of non negative integers, the following relations hold:*

(a) $$\mathcal{Z}_l^{(i,j)} \cap \mathcal{Z}_r^{(i,j)} = \mathcal{T}^{(i,j)} .$$

and

(b) $$\mathcal{Z}_l^{(i,j)} \vee \mathcal{Z}_r^{(i,j)} = (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i,j)}$$

Proof. We will first prove (a). Take any semigroup S in $\mathcal{Z}_l^{(i,j)} \cap \mathcal{Z}_r^{(i,j)}$, and take any elements $s = s_1 s_2 \ldots s_i$, $t = t_1 t_2 \ldots t_j$, x and y.

Then

$$sxt = (s_1 s_2 \ldots s_i)x (t_1 t_2 \ldots t_j) = (s_1 s_2 \ldots s_i)xy (t_1 t_2 \ldots t_j) \quad (\text{since } S \in \mathcal{Z}_l^{(i,j)})$$

$$= (s_1 s_2 \ldots s_i)y (t_1 t_2 \ldots t_j) = syt.$$

The second last equality holds since $S \in \mathcal{Z}_r^{(i,j)}$, and so $S \in \mathcal{T}^{(i,j)}$.

Conversely, take any semigroup S in $\mathcal{T}^{(i,j)}$. Now take any elements $s = s_1 s_2 \ldots s_i$, $t = t_1 t_2 \ldots t_j$, x and y in S.

Since

$$sxyt = (s_1 s_2 \ldots s_i)xy(t_1 t_2 \ldots t_j) = (s_1 s_2 \ldots s_i)x(t_1 t_2 \ldots t_j) = sxt$$

and

$$sxyt = (s_1 s_2 \ldots s_i)xy(t_1 t_2 \ldots t_j) = (s_1 s_2 \ldots s_i)y(t_1 t_2 \ldots t_j) = syt,$$

S is contained in both $\mathcal{Z}_l^{(i,j)}$ and $\mathcal{Z}_r^{(i,j)}$, and that proves (a).

Now, consider (b). First we observe that since both $\mathcal{Z}_l$ and $\mathcal{Z}_r$ are contained in $\mathcal{Z}_l \vee \mathcal{Z}_r$, by the containment preserving nature of the map $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$, both $\mathcal{Z}_l^{(i,j)}$ and $\mathcal{Z}_r^{(i,j)}$ are contained in $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i,j)}$. Hence,

$$\mathcal{Z}_l^{(i,j)} \vee \mathcal{Z}_r^{(i,j)} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i,j)}.$$

Conversely, consider the variety $\mathcal{Z}_l^{(0,1)} = \mathcal{Z}_r^{(1,0)} = [zxy = zy]$ of all inflations of rectangular bands. The containment $(\mathcal{Z}_l \vee \mathcal{Z}_r) \subseteq \mathcal{Z}_l^{(0,1)} = \mathcal{Z}_r^{(1,0)}$ is well known. The image of this under the maps $\mathcal{V} \mapsto \mathcal{V}^{(i,j-1)}$ and $\mathcal{V} \mapsto \mathcal{V}^{(i-1,j)}$ yield:

$$(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i-1,j)} \subseteq \mathcal{Z}_r^{(i,j)} \qquad \text{and} \qquad (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i,j-1)} \subseteq \mathcal{Z}_l^{(i,j)}.$$

Therefore, for any semigroup S in $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i,j)}$,

$$S/\theta(1,0) \in (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i-1,j)} \subseteq \mathcal{Z}_r^{(i,j)} \qquad \text{and} \qquad S/\theta(0,1) \in (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i,j-1)} \subseteq \mathcal{Z}_l^{(i,j)}.$$

Since $a \mapsto (a\theta(1,0), a\theta(0,1))$ is an embedding of S into $S/\theta(1,0) \times S/\theta(0,1)$, it follows that $S/\delta_1 \in \mathcal{Z}_l^{(i,j)} \vee \mathcal{Z}_r^{(i,j)}$, where $\delta_1 = \theta(1,0) \cap \theta(0,1)$. Using bisective arguments, as in Lemma 7:2.3, one can show that S is contained in the join $\mathcal{Z}_l^{(i,j)} \vee \mathcal{Z}_r^{(i,j)}$, and that completes the proof. ❑

Corollary 7:4.3 *For every ordered pair (i,j) of non negative integers, the following closed diamond exists:*

$$\begin{tikzcd}[column sep=small]
 & (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(i,j)} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathcal{Z}_l^{(i,j)} \arrow[dr, no head] & & \mathcal{Z}_r^{(i,j)} \arrow[dl, no head] \\
 & \mathcal{T}^{(i,j)} &
\end{tikzcd}$$

❑

We have thus described all relationships that exist among varieties of the types $\mathcal{Z}_l^{(i,j)}$ and $\mathcal{Z}_r^{(i,j)}$. In the last 4 sections, we have described the interrelationships that exist between the different types of varieties in $\wp$. In the remainder of this chapter, we will study lattices formed by structurally trivial varieties, making use of these recursive relations. In Section 7:5, we describe the skeleton of $\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^3)$, and building on that, in Section 7:6 we construct in an inductive manner, the skeleton of the general case $\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^n)$.

7:5 THE SKELETON OF THE LATTICE $\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^3)$

The lattice structure of $\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^2)$ was completely determined by Melnik (1971). Later, Petrich (1974) gave an alternative proof. (See Figure 0:4.15 for this lattice). In this section, we will describe the skeleton of $\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^3)$. This information will give us some idea of what to expect of the lattice structure of $\mathcal{L}_S((\mathcal{Z}_l \vee \mathcal{Z}_r)^n)$, as n increases, in view of the recursive varietal relations proved in the previous sections.

The next sequence of results re-expresses the equational classes appearing on Figure 0:4.15 in terms of the notation used in this chapter.

Lemma 7:5.1 $[abc = abxc, xy = (xy)^2] = (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(1,0)}$

Proof. Take any semigroup S from the equational class and take any elements e,f,g of S. Then ef = efef = eff = efgf. This proves that S satisfies the single identity that determines $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(1,0)}$ and so S is contained in that variety.

Conversely, take any S in $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(1,0)}$. Then $S/\theta(1,0)$ is a rectangular band. Since rectangular bands satisfy the identity xz = xyz, it follows that for any a,b,c,d $\in$ S, abc = abdc. To see that S also satisfies the identity $xy = (xy)^2$, take any a,b $\in$ S. Now, ab = a(bab) = (ab)(ab) = $(ab)^2$. Hence the equality holds. ❑

Lemma 7:5.2 $[abc = axbc, xy = (xy)^2] = (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,1)}$

Proof. This result is just the dual of Lemma 7:5.1. ❑

Lemma 7:5.3 $[abcd = abxcd, xy = (xy)^2] = (\mathcal{Z}_l \vee \mathcal{Z}_r)^2$

Proof. For any semigroup S in $(\mathcal{Z}_l \vee \mathcal{Z}_r)^2$ it is clear that S^2 is a rectangular band, and so it satisfies the identity $xy = (xy)^2$. Now, for any elements a,b,c,d,x in S we have

$$abcd = (ab)^2cd = ab(ab)cd = ab(abx)cd = (ab)^2x(cd) = abxcd.$$

This proves that S belongs to the equational class.

Conversely, for any semigroup S from the equational class, we have by Theorem 2:2.6 that $S/\theta(1,1)$ satisfies bc = bxc. To show that S belongs to $(Z_l \vee Z_r)^2$ we need only show that it satisfies xy = xy(wt)xy. By assumption, S satisfies $xy = (xy)^2$ and so for any elements a,b,c,d of S, we have

$$ab = (ab)\,[(ab)(ab)]\,(ab) = (ab)\,[(ab)(cd)(ab)]\,(ab) = (ab)^2\,(wt)(ab)^2 = (ab)(wt)(ab).$$

We have the second equality since $S/\theta(1,1)$ satisfies bc = bxc. This proves that S satisfies the identity which determines $(Z_l \vee Z_r)^2$, and that completes the proof. ❑

Lemma 7:5.4 $\qquad (Z_l \vee Z_r)^{(0,1)} \vee (Z_l \vee Z_r)^{(1,0)} = (Z_l \vee Z_r)^2$

Proof. This result is of course Lemma 7:2.3 for the case n = 1. We will give here an alternative proof which makes use of Figure 0:4.15, to show that our results are consistent with known results:

$$(Z_l \vee Z_r)^{(0,1)} \vee (Z_l \vee Z_r)^{(1,0)} = [abc = abxc,\ xy = (xy)^2] \vee [bcd = bxcd,\ xy = (xy)^2]$$

(from Lemma 7:5.1 and Lemma 7:5.2)

$$= [abcd = abxcd,\ xy = (xy)^2] \quad \text{(from Figure 0:4.15)}$$

$$= (Z_l \vee Z_r)^2 \qquad \text{(from Lemma 7:5.3).}$$ ❑

Lemma 7:5.5 $\qquad [zxywt = zxwt,\ zxxy = zxy] = (Z_l \vee Z_r)^{(1,1)}$

Proof. Consider first the equational class

$$[zxywt = zxwt,\ zxxy = zxy] = [zxywt = zxwt] \cap [zxxy = zxy]$$

$$= [xyw = xw]^{(1,1)} \cap [x^2 = x]^{(1,1)}$$

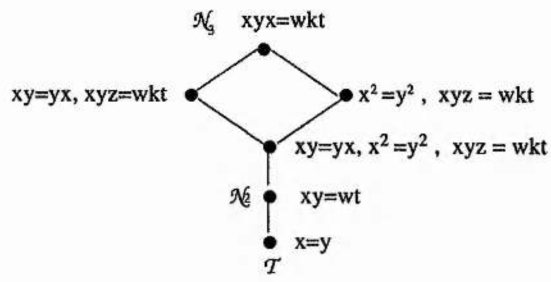
$$= \{\text{inflations of rectangular bands}\}^{(1,1)} \cap \{\text{bands}\}^{(1,1)}$$

$$= (\{\text{inflations of rectangular bands}\} \cap \{\text{bands}\})^{(1,1)}$$

$$= (\text{rectangular bands})^{(1,1)}$$

$$= (Z_l \vee Z_r)^{(1,1)}.$$ ❑

Theorem 7:5.6 *The following is the lattice of all 3-nilpotent semigroup varieties:*



Proof. To find all distinct varieties in the interval $[\mathcal{N}_2,\mathcal{N}_3]$, it suffices to consider the free 3-nilpotent semigroup on 2 generators $F_2(\mathcal{N}_3)$. This is because free 3-nilpotent semigroups on 3 or more generators will not be of help any more than $F_2(\mathcal{N}_3)$ can, since any word of length 3 or more is always zero. Now, the semigroup $F_2(\mathcal{N}_3)$ is finite and has the following Cayley table:

	a	b	aa	bb	ab	ba	0
a	aa	ab	0	0	0	0	0
b	ba	bb	0	0	0	0	0
aa	0	0	0	0	0	0	0
bb	0	0	0	0	0	0	0
ab	0	0	0	0	0	0	0
ba	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

There are exactly six non trivial identities that can be satisfied by this semigroup. These are obtained by determining all possible pairing of 2-letter words in S^2, where $S = F_2(\mathcal{N}_3)$:

$$xx = xy, \qquad xx = yy, \qquad xx = yx, \qquad xy = yy, \qquad xy = yx, \qquad yy = yx.$$

It turns out that all six identities reduce to either xy = yx or xx = yy. By way of an example, suppose that a 3-nilpotent semigroup T satisfies the identity xx = xy. Then for any elements a,b,c,d of T,

$$ab = aa = a0 = 0 = c0 = cc = cd;$$

and so T is a null semigroup (which satisfies the identity $xw = yz$). Similarly, one can show that any 3-nilpotent semigroup that satisfies any one of the following identities is null:

$$xx = yx, \qquad xy = yy, \qquad yy = yx.$$

This leaves us with the following two identities:

$$xx = yx, \qquad x^2 = y^2.$$

Now, within the variety determined by $xyz = wkt$, one can show that the equational classes $[xyz = wkt, x^2 = y^2]$ and $[xyz = wkt, xy = yx]$ are incomparable, in the sense that neither contains the other. But of course their meet is the variety $[xyz = wkt, xy = yx, x^2 = y^2]$. Thus we have determined the diamond part of the lattice in Theorem 7:5.7, and the remaining part of the lattice follows easily. ❑

Lemma 7:5.7 *The following containment are strict.*

$$(Z_l \vee Z_r)^{(0,1)} \subset (Z_l \vee Z_r)^2 \subset (Z_l \vee Z_r)^{(1,1)}.$$

Proof. We have the first strict containment by putting $n = 1$, $i = 0$, and $j = 1$ in Lemma 7:2.5 (a). The second one is obtained from Lemma 7:2.2 by putting $i = 0 = j$ and $n = 2$. ❑

The next counter example is an alternative proof for the distinctiveness of the varieties featuring in the second containment of Lemma 7:5.7, namely $(Z_l \vee Z_r)^2$ and $(Z_l \vee Z_r)^{(1,1)}$.

Counterexample 7:5.8 Let $L = \{\alpha,\beta\}$ be a 2-element left zero band, and consider the natural (left) enga-product on $S = L^{(1)} \times L$. More precisely, define a product $\otimes$ on S (as in Example 1:2.10) by: for any elements $i = (\nu, \mu)$ and $j = (\upsilon, \kappa)$ of $S = L^{(1)} \times L$,

$$i \otimes j = \begin{cases} (\kappa, \mu\kappa), & \text{if } \nu = 1, \\ (\nu\kappa, \mu\kappa), & \text{otherwise} \end{cases}$$

Then from Example 1:2.10, $(S,\otimes)$ forms a semigroup and $S/\theta(1,0)$ is isomorphic to L. Now, again construct another semigroup starting with

(S,⊗), this time using the dual method, by considering the Cartesian product

$$T = S \times S^{(1)} = (L^{(1)} \times L) \times (L^{(1)} \times L)^{(1)},$$

and defining the natural (right) enga-product ⊕ on T by: for any elements

$$a = [p, r] \text{ and } b = [q, s] \text{ of } T = S \times S^{(1)},$$

define

$$a \oplus b = \begin{cases} [p \otimes q, p], & \text{if } s = \mathbf{1}, \\ [p \otimes q, p \otimes s], & \text{otherwise} \end{cases}$$

Distinguish the adjoined identity elements of $S^{(1)}$ and $L^{(1)}$, respectively, by **1** and 1. Since $T/\theta(1,1)$ is isomorphic to the left zero band L, which is a left zero band, (T,⊕) satisfies the identity zxyxk = zxk by Theorem 2:2.6. We will show however, that it does not satisfy the identity wtxywt = wt; and hence the containment

$$(\mathcal{Z}_l \vee \mathcal{Z}_r)^2 \subset (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(1,1)}$$

(of Lemma 7:5.7) is strict.

Now, put $w = [(1,\alpha), \mathbf{1}]$, $t = [(\alpha,\beta), 1]$, $x = [(\alpha,\alpha), \mathbf{1}]$ and $y = [(\beta,\beta), \mathbf{1}]$.

Then

$$w \oplus t = [(1,\alpha), \mathbf{1}] \oplus [(\alpha,\beta), 1] = [(1,\alpha) \otimes (\alpha,\beta), (1,\alpha) \otimes \mathbf{1}] = [(\beta,\alpha), (1,\alpha)]$$

and

$$x \oplus y = [(\alpha,\alpha), \mathbf{1}] \oplus [(\beta,\beta), \mathbf{1}] = [(\alpha,\alpha) \otimes (\beta,\beta), (\alpha,\alpha) \otimes \mathbf{1}] = [(\alpha,\alpha), (\alpha,\alpha)].$$

Therefore,

$$\begin{aligned}(w \oplus t) \oplus (x \oplus y) \oplus (w \oplus t) &= [(\beta,\alpha),(1,\alpha)] \oplus [(\alpha,\alpha), (\alpha,\alpha)] \oplus [(\beta,\alpha), (1,\alpha)] \\ &= [(\beta,\alpha) \otimes (\alpha,\alpha) \otimes (\beta,\alpha), (\beta,\alpha) \otimes (\alpha,\alpha) \otimes (1,\alpha)] \\ &= [(\beta,\alpha), (\beta,\alpha)] \neq [(\beta,\alpha), (1,\alpha)] = w \oplus t;\end{aligned}$$

and thus the semigroup (T,⊕) does not satisfy the identity wtxywt = wt, which proves that $(\mathcal{Z}_l \vee \mathcal{Z}_r)^2 \neq (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(1,1)}$. ❑

On Figure 7:5.9 below, the skeleton of $\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^3)$ is shown to form a structure that looks somewhat like an inverted pyramid. Of course, many other varieties are missing from this figure (see Theorem 7:5.6), but it does show however that the skeleton $\mathcal{L}_S((Z_l \vee Z_r)^3)$ is a disjoint union of the three planes P(1), P(2) and P(3), each of which will be shown later to be closed under varietal joins and meets.

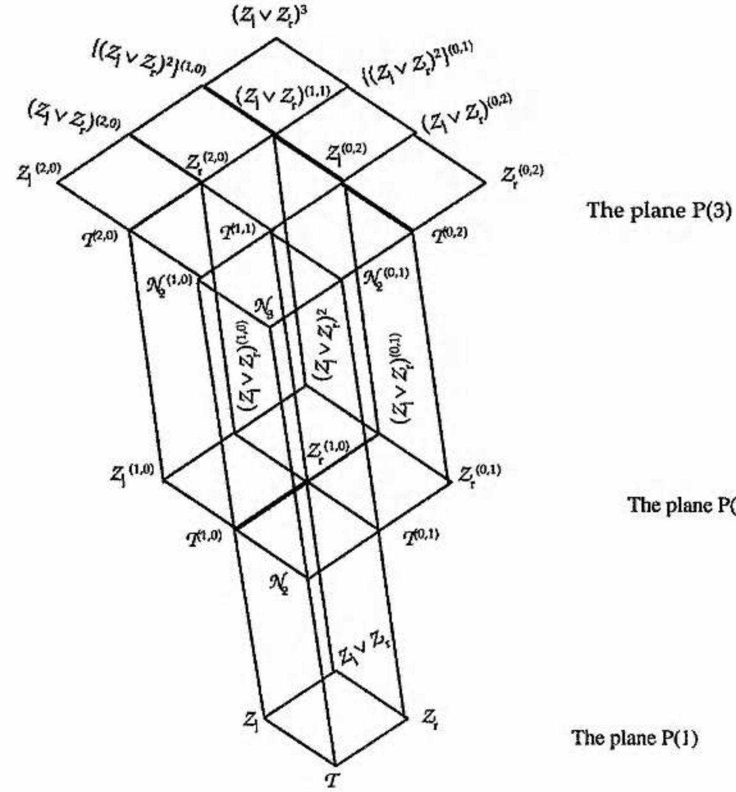


Figure 7:5.9 The skeleton of the lattice $\mathcal{L}((Z_l \vee Z_r)^3)$

Theorem 7:5.10 *The skeleton of the lattice* $\mathcal{L}((Z_l \vee Z_r)^3)$ *is shown on Figure 7:5.9 above.*

Proof. The lattice of all subvarieties of $(Z_l \vee Z_r)^2$ is given on Figure 0:4.15. Of course, its subvarieties are labelled alternatively on Figure 7:5.9 using our notation.

To prove this theorem, we need only show that each one of the diamonds occurring on the plane P(3), as shown in the above figure above, is *closed*. And secondly, we need to show that varieties on plane P(3) relate (in the sense of containments) to those on the plane P(2) as shown in the figure above.

Now, consider the varieties on plane P(3). There are exactly 9 diamonds featured on it. We will verify the existence and the closure of each diamond occurring there on the figure:

(1) By putting i = 0 = j and n = 2 in Corollary 7:2.8 (a) we have the following closed diamond:

$$\begin{tikzcd}[column sep=small]
& (Z_l \vee Z_r)^3 \arrow[dl, dash] \arrow[dr, dash] & \\
\{(Z_l \vee Z_r)^2\}^{(1,0)} \arrow[dr, dash] & & \{(Z_l \vee Z_r)^2\}^{(0,1)} \arrow[dl, dash] \\
& \{(Z_l \vee Z_r)\}^{(1,1)} &
\end{tikzcd}$$

(2) By putting i = 1 = j in Corollary 7:4.3 we have the following closed diamond:

$$\begin{tikzcd}[column sep=small]
& (Z_l \vee Z_r)^{(1,1)} \arrow[dl, dash] \arrow[dr, dash] & \\
Z_r^{(2,0)} = Z_l^{(1,1)} \arrow[dr, dash] & & Z_r^{(1,1)} = Z_l^{(0,2)} \arrow[dl, dash] \\
& \mathcal{T}^{(1,1)} &
\end{tikzcd}$$

(3) By putting i = 1 and j = 0 in Corollary 7:2.8 (b) we have the following closed diamond:

$$\begin{tikzcd}[column sep=small]
& \{(Z_l \vee Z_r)^2\}^{(1,0)} \arrow[dl, dash] \arrow[dr, dash] & \\
(Z_l \vee Z_r)^{(2,0)} \arrow[dr, dash] & & \{(Z_l \vee Z_r)\}^{(1,1)} \arrow[dl, dash] \\
& Z_r^{(2,0)} = Z_l^{(1,1)} &
\end{tikzcd}$$

(4) By putting i = 2 and j = 0 in Corollary 7:4.3 we have the following closed diamond:

$$\begin{tikzcd}[column sep=small]
& (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(2,0)} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{Z}_l^{(2,0)} \arrow[dr, dash] & & \mathcal{Z}_r^{(2,0)} = \mathcal{Z}_l^{(1,1)} \arrow[dl, dash] \\
& \mathcal{T}^{(2,0)} &
\end{tikzcd}$$

(5) By putting n = 1, i = 0 and j = 1 in Corollary 7:2.8 (b) we have the following closed diagram:

$$\begin{tikzcd}[column sep=small]
& \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^2\}^{(0,1)} \arrow[dl, dash] \arrow[dr, dash] & \\
\{(\mathcal{Z}_l \vee \mathcal{Z}_r)\}^{(1,1)} \arrow[dr, dash] & & (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,2)} \arrow[dl, dash] \\
& \mathcal{Z}_r^{(1,1)} = \mathcal{Z}_l^{(0,2)} &
\end{tikzcd}$$

(6) By putting i = 0 and j = 2 in Corollary 7:4.3 we have the following closed diamond:

$$\begin{tikzcd}[column sep=small]
& (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,2)} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{Z}_r^{(1,1)} = \mathcal{Z}_l^{(0,2)} \arrow[dr, dash] & & \mathcal{Z}_r^{(0,2)} \arrow[dl, dash] \\
& \mathcal{T}^{(0,2)} &
\end{tikzcd}$$

(7) By putting i = 1 and j = 0 in Corollary 7:3.5 (b) we have the following closed diamond:

$$\begin{tikzcd}[column sep=small]
& \mathcal{Z}_r^{(2,0)} = \mathcal{Z}_l^{(1,1)} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{T}^{(2,0)} \arrow[dr, dash] & & \mathcal{T}^{(1,1)} \arrow[dl, dash] \\
& \mathcal{N}_2^{(1,0)} &
\end{tikzcd}$$

(8) By putting i = 0 and j = 1 in Corollary 7:3.5 (b) we have the following closed diamond:

$$\begin{tikzcd}[column sep=small]
 & Z_r^{(1,1)} = Z_l^{(0,2)} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{T}^{(1,1)} \arrow[dr, dash] & & \mathcal{T}^{(0,2)} \arrow[dl, dash] \\
 & \mathcal{N}_2^{(0,1)} &
\end{tikzcd}$$

(9) By putting i = 0 = j and k = 2 in Corollary 7:3.5 (a) we have the following closed diamond:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{T}^{(1,1)} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{N}_2^{(1,0)} \arrow[dr, dash] & & \mathcal{N}_2^{(0,1)} \arrow[dl, dash] \\
 & \mathcal{N}_3 &
\end{tikzcd}$$

This completes the verification of the closure and the existence of every diamond on plane P(3), as shown on Figure 7:5.9. By putting together all the pieces of diamonds, superimposing parts which do coincide, we form the entire plane P(3). This same method will be used later, inductively, to construct other larger planes in the next section.

Our next task will be to determine the containment relationships that exist between varieties on P(3) and P(2). More precisely, we will show that each containment relationship shown in Figure 7:5.9, between varieties on P(3) and P(2), is indeed strict.

(a) The strict containment $(Z_l \vee Z_r)^2 \subset (Z_l \vee Z_r)^{(1,1)}$ is obtained by putting i = 0 = j and n = 2 in Lemma 7:2.2 (or, alternatively, from Lemma 7:5.9).

(b) The strict containment $(Z_l \vee Z_r)^{(0,1)} \subset Z_r^{(1,1)}$ holds, by observing that it is the image under the map $\mathcal{V} \mapsto \mathcal{V}^{(0,1)}$ of the known strict containment $(Z_l \vee Z_r) \subset Z_r^{(1,0)}$ (since $Z_r^{(1,0)}$ is the variety of all inflations of rectangular bands). Dually, $(Z_l \vee Z_r)^{(1,0)} \subset Z_l^{(1,1)} = Z_r^{(2,0)}$.

(c) We have the strict containment $Z_r^{(0,1)} \subset \mathcal{T}^{(0,2)}$, since it is the image under the mapping $\mathcal{V} \mapsto \mathcal{V}^{(0,1)}$ of the well known strict containment $Z_r \subset \mathcal{T}^{(0,1)}$ (see Figure 0:4.15); and by a dual argument we have $Z_r^{(1,0)} \subset \mathcal{T}^{(2,0)}$.

(d) And, of course, the containment $\mathcal{N}_2 \subset \mathcal{N}_3$ is strict.

(e) Since $\mathcal{T} \subset \mathcal{N}_2$ is a strict containment, we have that both the containments $\mathcal{T}^{(0,1)} \subset \mathcal{N}_2^{(0,1)}$ and $\mathcal{T}^{(1,0)} \subset \mathcal{N}_2^{(1,0)}$ are strict.

(f) Finally, since the containment $Z_1 \subset \mathcal{T}^{(1,0)}$ is strict, we have that the containment $Z_1^{(0,1)} \subset \mathcal{T}^{(1,1)}$ is strict.

We have thus shown that every containment relation between varieties on P(3) and P(2), as shown on Figure 7:5.9, does exist and that it is strict. ❑

The plane P(1) is formed entirely by varieties of rectangular bands; and P(2) is formed precisely by the lattice interval $[\mathcal{N}_2, (Z_1 \vee Z_r)^2]$, as shown on Figure 7:5.9. However, it is not clear at this stage whether or not the plane P(3) consists precisely of the varieties in the interval $[\mathcal{N}_3, (Z_1 \vee Z_r)^3]$. That is, we are not sure if P(3) is dense, in the sense that there may be other varieties in the interval $[\mathcal{N}_3, (Z_1 \vee Z_r)^3]$ which are not included in P(3).

7:6 THE SKELETON OF THE LATTICE $\mathcal{L}((Z_1 \vee Z_r)^n)$

From our observation of the skeleton of $\mathcal{L}((Z_1 \vee Z_r)^n)$ for the cases n = 1, 2 and 3, and together with the recursive relations proved in Sections 7:2. 7:3 and 7:4, it is natural to expect some kind of a repeated pattern as n increases. The lattices on Figure 0:4.15 and Figure 7:5.9 look somewhat like inverted pyramids. Our purpose in this section is to describe the lattice structure of the following partially ordered set :

$$\mathcal{L}_S((Z_1 \vee Z_r)^n) = \{\mathcal{V} \in \wp : \mathcal{V} \subseteq (Z_1 \vee Z_r)^n\}.$$

Equivalently, in terms of our terminology, this lattice (see Figure 7:7.4) is just the skeleton of $\mathcal{L}((Z_1 \vee Z_r)^n)$. For each positive integer n, we will call the following set P(n) of varieties, the *n-th plane:*

$$P(n) = \{Z \in \wp : \mathcal{N}_n \subseteq Z \subseteq (Z_1 \vee Z_r)^n\}.$$

The lattice diagram of the first five planes P(1) - P(5) are given in Example 7:7.7. In fact, we met P(1), P(2) and P(3) earlier in Figure 7:5.9. In general, the plane P(n) forms the skeleton of the lattice interval $[\mathcal{N}_n, (Z_1 \vee Z_r)^n]$, and its lattice structure is given on Figure 7:6.1. That lattice structure looks somewhat superficially like a plane. We will see that the varieties which sit on P(n) consists precisely of the following:

(I) $\{ \mathcal{N}_k^{(i,j)} : i+j+k = n \}$

(II) $\{ Z_1^{(i,j)} : i+j+1 = n \}$

(III) $\{ Z_r^{(i,j)} : i+j+1 = n \}$

(IV) $\{\{(Z_1 \vee Z_r)^k \}^{(i,j)} : i+j+k = n \}$

To verify that each one of the above varieties do really sit on P(n), we need only show that each one of them lies somewhere between $\mathcal{N}_n$ and $(Z_1 \vee Z_r)^n$. The details of this verification are routine and are therefore left out. And conversely, to show that the varieties in (I) - (IV) above are precisely all of them, that is, that there are no other varieties on P(n), take any variety $\mathcal{A}$ on P(n), and suppose that $\mathcal{A} = \mathcal{N}_k^{(i,j)}$, (say, of type (7:1.20)).
Then

$$\mathcal{N}_n \subseteq \mathcal{A} \subseteq (Z_1 \vee Z_r)^n.$$

By Lemma 0:4.21, the first containment implies that $n \leq k+i+j$, and the second containment implies $k+i+j \leq n$. These inequalities, together, imply that $k+i+j = n$. We have thus shown that if a variety of type (7:1.20) exists on P(n), then it is necessarily of type (I) above. Similarly, it can be shown that if $\mathcal{A}$ is assumed to be from one of the other remaining types (7:1.17) - (7:1.19), then $\mathcal{A}$ is from one of the types in the list (II), (III) or (IV) given above.

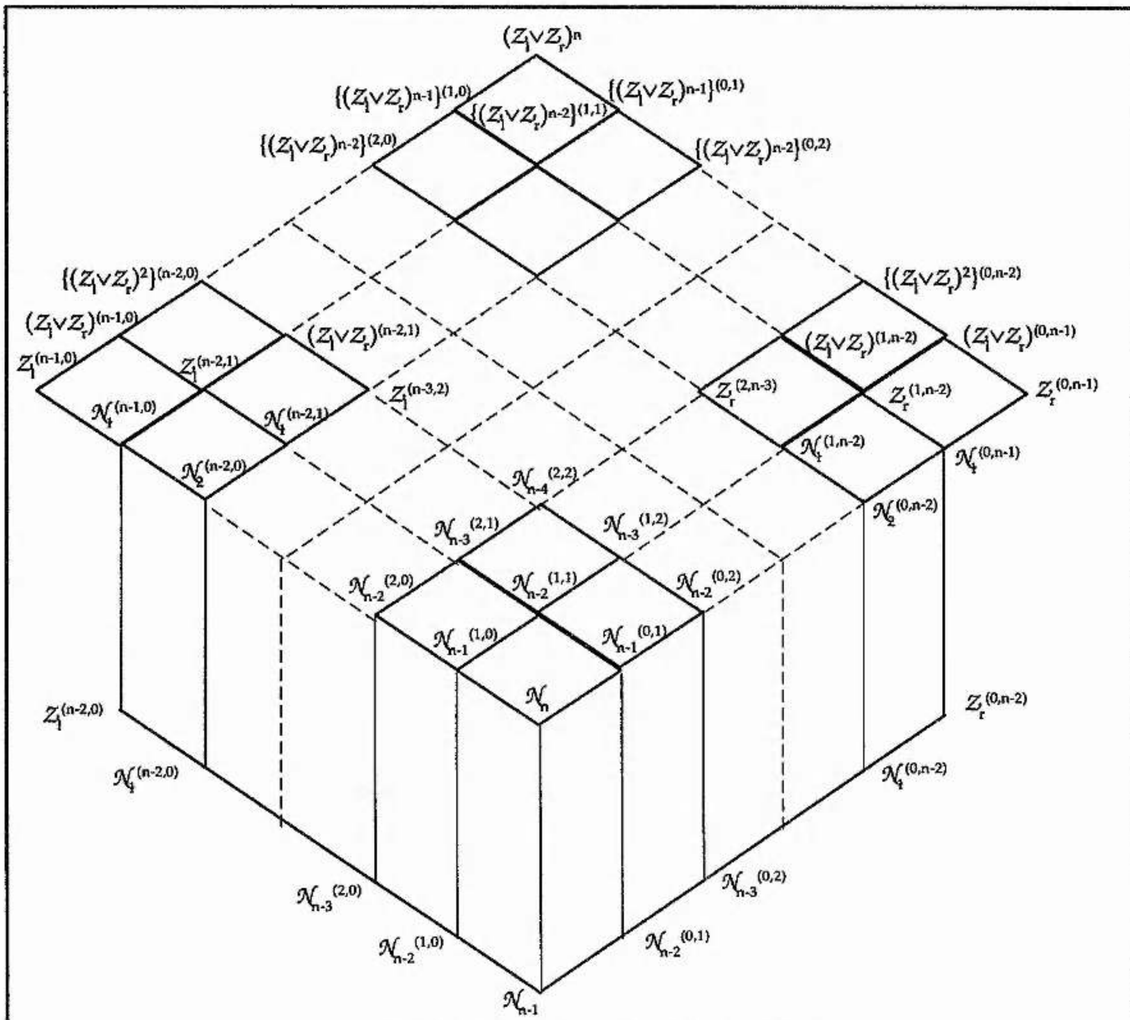


Figure 7:6.1 The plane P(n) as it relates to the plane P(n-1).
(see Example 7:7.7 for some particular examples of these planes)

The above figure is obtained by putting together all the diamonds and recursive relations proved in the earlier sections of the present chapter. One can prove each of the diamonds on P(n) using Corollary 7:2.8, Corollary 7:4.3 and Corollary 7:3.5 as we did in the proof of Theorem 7:5.10, where we proved the existence and the closure of each diamond on P(3), and then putting the diamonds together, superimposing those edges and vertices that coincide. At this stage, this lattice diagram describes merely the containment relationships between members of the partially ordered set P(n). However, we will show later that P(n) is closed under varietal joins and meets, and hence its lattice structure is precisely as shown on the above figure.

Any variety on P(n) which is a subvariety of either $\mathcal{Z}_l^{(n-1,0)}$ or $\mathcal{Z}_r^{(0,n-1)}$ will be called a *P(n)-join irreducible variety.* If $\mathcal{A} \in$ P(n) and is not P(n)-join irreducible, then it can be expressed as a join of two other distinct varieties on P(n). This means that the varieties in the following list are not P(n)-join irreducible:

Case (I): If $\mathcal{A} = \mathcal{N}_k^{(i,j)}$, where $i+j+k = n$, $1 \le i \le n-1$ and $1 \le j \le n-1$, then

$\mathcal{N}_k^{(i,j)} = \mathcal{N}_{k+1}^{(i-1,j)} \vee \mathcal{N}_{k+1}^{(i,j-1)}$, for $k = 2,3,4,\dots$ (by Theorem 7:3.3).

Case (II): If $\mathcal{A} = \mathcal{Z}_l^{(i,j)}$, where $i+j+1 = n$, $0 \le i \le n-1$ and $1 \le j \le n-1$, then

$\mathcal{Z}_l^{(i,j)} = \mathcal{N}_1^{(i,j)} \vee \mathcal{N}_1^{(i+1,j-1)}$ (by Theorem 7:3.3)

Case (III): If $\mathcal{A} = \mathcal{Z}_r^{(i,j)}$, where $i+j+1 = n$, $1 \le i \le n-1$ and $0 \le j \le n-1$, then

$\mathcal{Z}_r^{(i,j)} = \mathcal{N}_1^{(i,j)} \vee \mathcal{N}_1^{(i-1,j+1)}$ (by Theorem 7:3.3)

Case (IV): If $\mathcal{A} = \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(i,j)}$, where $i+j+k = n$, $0 \le i \le n-1$, $0 \le j \le n-1$ and $1 \le k \le n-1$, then from Theorem 7:2.6 we have:

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(i,j)} = \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{k-1}\}^{(i-1,j)} \vee \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{k-1}\}^{(i,j-1)}.$$

If $\mathcal{A}$ is, however, a subvariety of either $\mathcal{Z}_r^{(0,n-1)}$ or $\mathcal{Z}_l^{(n-1,0)}$, which are the only cases omitted in the above list (I) - (IV), then it is P(n)-join irreducible. In fact, as can be observed from Figure 7:6.1, the P(n)-join irreducible varieties are precisely the following:

$$\{\mathcal{N}_{n-i}{}^{(i,0)},\ \mathcal{N}_{n-j}{}^{(0,j)},\ \mathcal{Z}_l{}^{(n-1,0)},\ \mathcal{Z}_r{}^{(0,n-1)} :\ 0 \le i \le n\text{-}1 \text{ and } \ 0 \le j \le n\text{-}1\}.$$

A P(n)-join irreducible variety cannot be expressed as a join of two other distinct varieties both of which are on P(n). To verify that the above set is precisely all of them, take any P(n)-join irreducible variety $\mathcal{Q}$ on P(n), and suppose that $\mathcal{Q}$ is a subvariety of $\mathcal{Z}_l{}^{(n-1,0)}$ on $\wp$. By working down the list of possible candidates in the list (7:1.17) - (7:1.20), which form $\wp$, we can at once rule out the types $\mathcal{Z}_r{}^{(i,j)}$ and $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(i,j)}$. This is because the variety $\mathcal{Z}_l{}^{(n-1,0)}$ does not contain any right zero band, but clearly, both $\mathcal{Z}_r{}^{(i,j)}$ and $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(i,j)}$ do contain such bands.

Now, suppose that $\mathcal{Q}$ is of the type $\mathcal{Z}_l{}^{(i,j)}$. Then since i+j+1 = n, the extra condition that j = 0 implies i = n-1, and so $\mathcal{Q} = \mathcal{Z}^{(n-1,0)}$.

The other possibility is where $\mathcal{Q} = \mathcal{N}_k{}^{(i,j)}$. Again, in this case, since $\mathcal{Q}$ is a subvariety of $\mathcal{Z}_l{}^{(n-1,0)}$, we must have j = 0. But since i+j+k = n, we must have k = n-j and it follows therefore, that $\mathcal{Q} = \mathcal{N}_{n-i}{}^{(i,0)}$. Thus, it follows that the subvarieties of $\mathcal{Z}_l{}^{(n-1,0)}$ includes $\mathcal{Z}_l{}^{(n-1,0)}$ itself and the following varieties:

$$\mathcal{N}_{n-i}{}^{(i,0)} \text{ where } i = 0,1,2,\ldots,n\text{-}1\,.$$

Dually, the P(n)-join irreducible subvarieties of $\mathcal{Z}_r{}^{(0,n-1)}$ includes $\mathcal{Z}_r{}^{(0,n-1)}$ itself and the members of the following family:

$$\mathcal{N}_{n-j}{}^{(0,j)} \text{ where } j = 0,1,2,\ldots,n\text{-}1\,.$$

The subvarieties of $\mathcal{Z}_l{}^{(n-1,0)}$ form the following sequence of containments (see Figure 7:6.1):

(®) $$\mathcal{N}_n \subseteq \mathcal{N}_{n-1}{}^{(1,0)} \subseteq \mathcal{N}_{n-2}{}^{(2,0)} \subseteq \mathcal{N}_{n-3}{}^{(3,0)} \subseteq \ldots \subseteq \mathcal{N}_1{}^{(n-1,0)} \subseteq \mathcal{Z}_l{}^{(n-1,0)}\,.$$

and

(®') $$\mathcal{N}_n \subseteq \mathcal{N}_{n-1}{}^{(0,1)} \subseteq \mathcal{N}_{n-2}{}^{(0,2)} \subseteq \mathcal{N}_{n-3}{}^{(0,3)} \subseteq \ldots \subseteq \mathcal{N}_1{}^{(0,n-1)} \subseteq \mathcal{Z}_r{}^{(0,n-1)}.$$

Lemma 7:6.2 *For every $n \ge 1$ and any $\mathcal{A} \in$ P(n),*

$$\mathcal{A} = (\mathcal{A} \cap \mathcal{Z}_l{}^{(n-1,0)}) \vee (\mathcal{A} \cap \mathcal{Z}_r{}^{(0,n-1)}).$$

Proof. In the preceding discussion, we have determined all P(n)-join irreducible varieties, and have shown them to form the two chains (®) and (®'); and, we have also determined all the varieties which are not P(n)-join irreducible on P(n).

Suppose $\mathcal{A} \in P(n)$ and that it is not P(n)-join irreducible. Then from the list above, there exist varieties A and A′ on P(n) such that $\mathcal{A} = A \vee A'$. If A and A′ are subvarieties of either $Z_r^{(0,n-1)}$ or $Z_r^{(n-1,0)}$, in which case they are P(n)-join irreducible, there is nothing further to prove. Suppose they are not. Then there exists varieties B, B′ and C, C′ on P(n) such that $A = B \vee B'$, $A' = C \vee C'$ and $\mathcal{A} = B \vee B' \vee C \vee C'$.

Repeating this process on B, B′, C, and C′, expressing each variety as a join of other subvarieties of P(n), wherever possible. At any stage if a subvariety of either $Z_r^{(0,n-1)}$ or $Z_l^{(n-1,0)}$ is obtained, then put it aside, and continue the process on the remaining. After finitely many repetitions, the process terminates, and by then $\mathcal{A}$ is expressed completely as a join of P(n)-join irreducible varieties on P(n).

In other words, there exist subvarieties $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_t$ of $Z_l^{(n-1,0)}$ and subvarieties $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_w$ of $Z_r^{(0,n-1)}$ such that

$$\mathcal{A} = (\mathcal{X}_1 \vee \mathcal{X}_2 \vee \mathcal{X}_3 \vee \ldots \vee \mathcal{X}_t) \vee (\mathcal{Y}_1 \vee \mathcal{Y}_2 \vee \mathcal{Y}_3 \vee \ldots \vee \mathcal{Y}_w)$$

$$= \mathcal{X} \vee \mathcal{Y}$$

(where $\mathcal{X} = \max(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_t)$ and $\mathcal{Y} = \max(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_w)$)

$$= (\mathcal{A} \cap Z_l^{(n-1,0)}) \vee (\mathcal{A} \cap Z_r^{(0,n-1)}). \qquad \square$$

Corollary 7:6.3 *Every variety $\mathcal{A}$ on the plane P(n) can be uniquely expressed as a join of P(n)-join irreducible varieties on P(n).*
That is

$$\mathcal{A} = \mathcal{X} \vee \mathcal{Y}, \qquad \textit{where } \mathcal{X} \subseteq Z_l^{(n-1,0)} \textit{ and } \mathcal{Y} \subseteq Z_r^{(0,n-1)}.$$

In particular, we have

$$(Z_l \vee Z_r)^n = Z_l^{(n-1,0)} \vee Z_r^{(0,n-1)}.$$

Proof. For any variety $\mathcal{A}$ on P(n), in view of the proof of Lemma 7:6.2, it suffices to show that both $\mathcal{A} \cap Z_l^{(n-1,0)}$ and $\mathcal{A} \cap Z_r^{(0,n-1)}$ belong to P(n). In fact, in Lemma 7:6.4, we show that $\mathcal{A} \cap Z_l^{(n-1,0)} \in P(n)$, and by duality, we also have $\mathcal{A} \cap Z_r^{(0,n-1)} \in P(n)$. $\square$

It is tedious but necessary for us to verify each one of the equalities given in Lemma 7:6.4 below if we are to arrive at our desired conclusion in the above proof.

Lemma 7:6.4 *The following equalities hold on P(n) :*

(a) *For every variety* $\mathcal{A}$ *on* P(n) *such that*

$\mathcal{N}_{n-i}^{(i,0)} \subseteq \mathcal{A} \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$, *we have* $\mathcal{A} \cap \mathcal{Z}_l^{(n-1,0)} = \mathcal{N}_{n-i}^{(i,0)}$

(b) *For every variety* $\mathcal{A}$ *on* P(n) *such that*

$\mathcal{Z}_l^{(n-1,0)} \subseteq \mathcal{A} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^n$, *we have that* $\mathcal{A} \cap \mathcal{Z}_l^{(n-1,0)} = \mathcal{Z}_l^{(n-1,0)}$.

Proof. First, we observe that every variety $\mathcal{A}$ on P(n) occurs in one of the intervals given in (a) or (b). The statement (b) holds simply because $\mathcal{Z}_l^{(n-1,0)} \subseteq \mathcal{A}$. To prove (a), we will first prove the following particular cases:

(@) $\qquad \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)} \cap \mathcal{Z}_l^{(n-1,0)} = \mathcal{N}_{n-i}^{(i,0)}$, where $i = 1, 2, \ldots, n$.

Take any S in $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)} \cap \mathcal{Z}_l^{(n-1,0)}$, and any elements $a = a_1a_2 \ldots a_i$, $b = b_1b_2 \ldots b_{n-i}$, and $c = c_1c_2 \ldots c_{n-i}$ of S.
Then

$$ab = (a_1a_2 \ldots a_i)(b_1b_2 \ldots b_{n-i})$$

$$= [(a_1a_2 \ldots a_i)(c_1c_2 \ldots c_{n-i})(a_1a_2 \ldots a_i)](b_1b_2 \ldots b_{n-i})$$

(since S belongs to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$)

$$= a\,(c_1c_2 \ldots c_{n-i-1})(c_{n-i})(ab)$$

$$= a\,(c_1c_2 \ldots c_{n-i-1})(c_{n-i}) = ac.$$

The second last equality holds since S belongs to $\mathcal{Z}_l^{(n-1,0)}$, and hence S belongs to $\mathcal{N}_{n-i}^{(i,0)}$.

To prove the reverse containment of (@), take any semigroup S in $\mathcal{N}_{n-i}^{(i,0)}$. We will first show that S belongs to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$. Take any elements $a = a_1a_2 \ldots a_i$, $b = b_1b_2 \ldots b_{n-i}$, and $c = c_1c_2 \ldots c_{n-i}$ of S.
Then

$$ab = (a_1a_2 \ldots a_i)[b_1b_2 \ldots b_{n-i}]$$

$$= (a_1a_2 \ldots a_i)[(b_1b_2 \ldots b_{n-i})(c_1c_2 \ldots c_{n-i})(b_1b_2 \ldots b_{n-i})]$$

$$= abcb.$$

We have the second equality, since S belongs to $\mathcal{N}_{n-i}^{(i,0)}$. Hence S belongs to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$. To show that S belongs also to $\mathcal{Z}_l^{(n-1,0)}$, take any element $s = s_1 \ldots s_{n-1}$, x and y. Now,

$$sx = (s_1s_2 \ldots s_{n-1})x = (s_1s_2 \ldots s_i)(s_{i+1}s_{i+2} \ldots s_{n-1}x)$$
$$= (s_1s_2 \ldots s_i)(s_{i+1}s_{i+2} \ldots s_{n-1}xy) = sxy.$$

The third equality holds since S belongs to $\mathcal{N}_{n-i}^{(i,0)}$, and hence the equality (@) holds. To prove the general case, we first observe that the following equality holds:

(%) $\qquad \mathcal{N}_{n-i}^{(i,0)} \cap \mathcal{Z}_1^{(n-1,0)} = \mathcal{N}_{n-i}^{(i,0)}, \qquad i = 0,1,\ldots,n-1$

and (%) holds because $\mathcal{N}_{n-i}^{(i,0)} \subseteq \mathcal{Z}_1^{(n-1,0)}$ (see the sequence (®)). This implies that for any variety $\mathcal{A}$ such that

$$\mathcal{N}_{n-i}^{(i,0)} \subseteq \mathcal{A} \subseteq \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$$

we have

$$\mathcal{N}_{n-i}^{(i,0)} = \mathcal{N}_{n-i}^{(i,0)} \cap \mathcal{Z}_1^{(n-1,0)}$$
$$\subseteq \mathcal{A} \cap \mathcal{Z}_1^{(n-1,0)} \subseteq \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^i\}^{(0,n-i)} \cap \mathcal{Z}_1^{(n-1,0)} = \mathcal{N}_{n-i}^{(i,0)}.$$

We have the first equality from (%), the next two containments are due to the interval in which $\mathcal{A}$ is taken from, and the last equality is precisely (@). That completes the proof for (a). ❑

Theorem 7:6.5 *The plane* P(n) *is closed under taking varietal joins.*

Proof. Take any $\mathcal{A}, \mathcal{B} \in$ P(n). By Corollary 7:6.3, we may express these varieties uniquely as a join of P(n)-join irreducible varieties as follows:

$$\mathcal{A} = \mathcal{X} \vee \mathcal{Y} \qquad \text{and} \quad \mathcal{B} = \mathcal{P} \vee \mathcal{Q}.$$

Therefore,

$$\mathcal{A} \vee \mathcal{B} = (\mathcal{X} \vee \mathcal{Y}) \vee (\mathcal{P} \vee \mathcal{Q}) = (\mathcal{X} \vee \mathcal{P}) \vee (\mathcal{Y} \vee \mathcal{Q}).$$

Now, from the chain formations by P(n)-join irreducible varieties (can be seen on Figure 7:6.1), the following equalities hold:

$$\mathcal{X} \vee \mathcal{P} = \max\{\mathcal{X}, \mathcal{P}\} \qquad \text{and} \quad \mathcal{Y} \vee \mathcal{Q} = \max\{\mathcal{Y}, \mathcal{Q}\}.$$

Hence, every varietal join on P(n) can be expressed as a join of some P(n)-join irreducible varieties. Thus, to prove the closure of P(n) under varietal joins, it suffices to show the closure of P(n) under the joins of P(n)-join irreducible varieties. Hence, every join on P(n) can be expressed as one of the following types of join: $\mathcal{N}_k^{(i,0)} \vee \mathcal{N}_t^{(0,j)}$, $\mathcal{N}_k^{(i,0)} \vee \mathcal{Z}_r^{(0,n-1)}$, $\mathcal{Z}_r^{(0,n-1)} \vee \mathcal{Z}_1^{(n-1,0)}$, or, $\mathcal{N}_k^{(0,j)} \vee \mathcal{Z}_1^{(n-1,0)}$.

In fact, the following equalities hold:

$$
\text{(a')}\qquad \mathcal{N}_{n-i}^{(i,0)} \vee \mathcal{N}_{n-j}^{(0,j)} = \begin{cases} \mathcal{Z}_l^{(i-1,j)} , & \text{if } i+j = n \\ \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{i+j-n}\}^{(n-j,n-i)} , & \text{if } i+j > n \\ \mathcal{N}_{n-(i+j)}^{(i,j)} , & \text{if } i+j < n \end{cases}
$$

$$
\text{(b')}\qquad \mathcal{N}_{n-i}^{(i,0)} \vee \mathcal{Z}_r^{(0,n-1)} = \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{i}\}^{(0,n-i)} \text{ for } i = 0,1,\ldots, n-1
$$

$$
\text{(c')}\qquad \mathcal{N}_{n-j}^{(0,j)} \vee \mathcal{Z}_l^{(n-1,0)} = \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{j}\}^{(0,n-j)} \text{ for } i = 0,1,\ldots, n-1
$$

$$
\text{(d')}\qquad \mathcal{Z}_r^{(0,n-1)} \vee \mathcal{Z}_l^{(n-1,0)} = (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}.
$$

The way to prove each of these equalities (a') - (d') is to start with the right hand side of each, and by following the procedure set out in the proof of Lemma 7:6.4, express these varieties as joins of some P(n)-join irreducible varieties on P(n). Then one arrives, finally, at the expression on the left. This leads us to the conclusion that $\mathcal{A} \vee \mathcal{B} \in$ P(n) for every $\mathcal{A}$ and $\mathcal{B}$ on P(n), and hence P(n) is closed under taking varietal joins. ❑

Next, we will show that P(n) is closed under varietal meets. To achieve this, we will proceed in a way somewhat dual to the approach taken in the proof of Theorem 7:6.5.

We will say a variety is *P(n)-meet irreducible* if it belongs to $\wp$ and exists in any one of the following two intervals:

$$
\mathcal{Z}_l^{(n-1,0)} \subseteq \mathcal{V} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}
$$

and

$$
\mathcal{Z}_r^{(0,n-1)} \subseteq \mathcal{V} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}.
$$

In fact, as can be observed on Figure 7:6.1, the P(n)-meet irreducible varieties on P(n) form the following chains, respectively:

$$
\mathcal{Z}_l^{(n-1,0)} \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{1}\}^{(n-1,0)} \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{2}\}^{(n-2,0)} \subseteq \ldots \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(1,0)} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}
$$

and

$$
\mathcal{Z}_r^{(0,n-1)} \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{1}\}^{(0,n-1)} \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{2}\}^{(0,n-2)} \subseteq \ldots \subseteq \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{n-1}\}^{(0,1)} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^{n}.
$$

Such varieties are those which cannot be expressed as a meet of two other distinct varieties on P(n).

Theorem 7:6.6 *The plane* P(n) *is also closed under taking varietal meets.*

Proof. First we will verify that there are no other P(n)-meet irreducible varieties other than those forming the above two chains. Suppose that $\mathcal{V}$ is P(n)-meet irreducible. Then, proceeding down through the list of all possible candidates given in (7:1.17) - (7:1.20), we can at once rule out the possibility of $\mathcal{V}$ being a variety of the type $\mathcal{N}_k^{(i,j)}$, since any variety of this type can be expressed as a meet of two other varieties on P(n) as shown in Theorem 7:4.2 and Theorem 7:3.4, namely:

$$\mathcal{N}_k^{(i,j)} = \begin{cases} \mathcal{Z}_l^{(i,j)} \cap \mathcal{Z}_r^{(i,j)} & , \ k=1 \\ \mathcal{N}_{k-1}^{(i+1,j)} \cap \mathcal{N}_{k-1}^{(i,j+1)} & , \text{ otherwise} \end{cases}$$

We also see that $\mathcal{V}$ cannot, for certain choices of k,i and j, be of the type $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(i,j)}$ since by Theorem 7:2.7 we have:

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(i,j)} = \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{k+1}\}^{(i-1,j)} \cap \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^{k+1}\}^{(i,j-1)}$$

where $k \geq 1$, and $k+i+j = n$ whenever $i \neq 0$ or $j \neq 0$ or both $i \neq 0$ and $j \neq 0$.

This means, however, that $\mathcal{V}$ could be:

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n-k,0)} \text{ and } \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n-k,0)} \text{ for } k = 1,2,\ldots\ldots n.$$

Furthermore, the following are not P(n)-meet irreducible since by Theorem 7:2.7 we have:

$$\mathcal{Z}_r^{(i,j)} = \{(\mathcal{Z}_l \vee \mathcal{Z}_r)\}^{(i+1,j)} \cap \{(\mathcal{Z}_l \vee \mathcal{Z}_r)\}^{(i,j+1)} \text{ for } i \geq 1 \text{ and } j \geq 0.$$

A variety of the type $\mathcal{Z}_r^{(i,j)}$ can be P(n)-meet irreducible only when $i = 0$, and in particular, $\mathcal{Z}_r^{(0,j)}$. But since $j+0+1 = n$, this means $\mathcal{Z}_r^{(0,n-1)}$ is the only meet irreducible variety of this type (7:1.19). Dually, $\mathcal{Z}_l^{(n-1,0)}$ is the only P(n)-meet irreducible variety of the type $\mathcal{Z}_l^{(i,j)}$.

We will show that every variety $\mathcal{A}$ on P(n) can be expressed as a meet of P(n)-meet irreducible varieties. In fact we have by Theorem 7:6.5 that both $\mathcal{A} \vee \mathcal{Z}_r^{(0,n-1)}$ and $\mathcal{A} \vee \mathcal{Z}_l^{(n-1,0)}$ lie on P(n), and are P(n)-meet irreducible from Figure 7:6.1.

Now, we will prove that

$$\mathcal{A} = (\mathcal{A} \vee \mathcal{Z}_l^{(n-1,0)}) \cap (\mathcal{A} \vee \mathcal{Z}_r^{(0,n-1)})$$

Suppose that $\mathcal{A}$ is not P(n)-meet irreducible. Then from Figure 7:6.1, and the list of all varieties which are not P(n)-meet irreducible, there exist varieties A and A′ on P(n) such that $\mathcal{A} = A \cap A'$. If both A and A′ are P(n)-meet irreducible, then there is nothing further to prove. Suppose they are not. Then there exist, varieties B, B′ and C, C′ on P(n) such that $A = B \cap B'$, $A' = C \cap C'$ and

$$\mathcal{A} = B \cap B' \cap C \cap C'.$$

Repeating this process on B, B′, C, and C′, expressing each variety as a meet of even other subvarieties of P(n), wherever possible. At any stage if a P(n)-meet irreducible variety is obtained, then put it aside, and continue the process on the remaining. After finitely many repetitions, this process terminates, and by then $\mathcal{A}$ is expressed completely as a meet of some P(n)-meet irreducible varieties on P(n). In other words, there exist varieties $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_t$ in the interval $Z_1^{(n-1,0)} \subseteq \mathcal{V} \subseteq (Z_1 \vee Z_r)^n$ and varieties $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_w$ in the interval $Z_r^{(0,n-1)} \subseteq \mathcal{V} \subseteq (Z_1 \vee Z_r)^n$ such that:

$$\mathcal{A} = (\mathcal{X}_1 \cap \mathcal{X}_2 \cap \mathcal{X}_3 \cap \ldots \cap \mathcal{X}_t) \cap (\mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \ldots \cap \mathcal{Y}_w)$$

$$= \mathcal{X} \cap \mathcal{Y} = (\mathcal{A} \vee Z_1^{(n-1,0)}) \cap (\mathcal{A} \vee Z_r^{(0,n-1)}),$$

where

$$\mathcal{X} = \min(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_t) = \mathcal{A} \vee Z_1^{(n-1,0)}$$

and

$$\mathcal{Y} = \min(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_w) = \mathcal{A} \vee Z_r^{(0,n-1)}.$$

We have thus proved that every variety on P(n) can be expressed uniquely as a meet of some P(n)-meet irreducible varieties.

Finally, to prove the closure of P(n) under varietal meets, take any varieties $\mathcal{A}, \mathcal{B} \in$ P(n). Then by what we have proved above, we may express these varieties uniquely as meets of some P(n)-meet irreducible varieties on P(n) as follows:

$$\mathcal{A} = \mathcal{X} \cap \mathcal{Y} \qquad \text{and} \quad \mathcal{B} = \mathcal{P} \cap \mathcal{Q}.$$

Therefore,

$$\mathcal{A} \cap \mathcal{B} = (\mathcal{X} \cap \mathcal{Y}) \cap (\mathcal{P} \cap \mathcal{Q}) = (\mathcal{X} \cap \mathcal{P}) \cap (\mathcal{Y} \cap \mathcal{Q}).$$

Now, since every meet on P(n) can be expressed as a meet of some P(n)-meet irreducible varieties, it suffices to show that P(n) is closed under taking meets of P(n)-meet irreducible varieties. In fact, every meet on P(n) reduces to a meet of the types given in Lemma 7:6.7. ❑

Again, it will be tedious but necessary to prove the following set of equalities, since by proving them we will be completing the proof of Theorem 7:6.6.

Lemma 7:6.7 *For each positive integer $n\geq 1$, the following equalities hold on* P(n)

(a") $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^k\}^{(0,n-k)} \cap \mathcal{Z}_1^{(n-1,0)} = \mathcal{N}_{n-k}^{(k,0)}$ for $k= 1, 2, \ldots, n$

(b") $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^k\}^{(n-k,0)} \cap \mathcal{Z}_r^{(0,n-1)} = \mathcal{N}_{n-k}^{(0,k)}$ for $k= 1, 2, \ldots, n$

(c") $\mathcal{Z}_r^{(0,n-1)} \cap \mathcal{Z}_1^{(n-1,0)} = \mathcal{N}_n$

(d") *For* $k = 1, 2, \ldots, n$ *and* $i = 1,2, \ldots, n$,

$$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^k\}^{(n-k,0)} \cap \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^i\}^{(0,n-i)} = \begin{cases} \mathcal{Z}_1^{(i-1,k)}, & \text{if } i+k = n \\ \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{i+k-n}\}^{(n-k,n-i)}, & \text{if } i+k > n \\ \mathcal{N}_{n-(i+k)}^{(i,k)}, & \text{if } i+k < n \end{cases}$$

Proof. The same proof given for (@) in the proof of Lemma 7:6.4 (a) works even for (a") above. And of course, (b") is just the dual of (a") and so that too holds. We therefore need to prove only (c") and (d").

(c") Now, to prove the equality $\mathcal{Z}_r^{(0,n-1)} \cap \mathcal{Z}_1^{(n-1,0)} = \mathcal{N}_n$, take any semigroup S in $\mathcal{Z}_r^{(0,n-1)} \cap \mathcal{Z}_1^{(n-1,0)}$. We will show that S belongs to $\mathcal{N}_n$. Take any elements $a_1 = a_1a_2 \ldots a_n$ and $b = b_1b_2 \ldots b_n$.

Then

$$a = (a_1a_2 \ldots a_{n-1})(a_n) = (a_1a_2 \ldots a_{n-1})(a_n)(b_1b_2 \ldots b_n) \quad (\text{since } S\in \mathcal{Z}_1^{(n-1,0)})$$

$$= (a_1a_2 \ldots a_n)(b_1)(b_2 \ldots b_n) = (b_1)(b_2 \ldots b_n) \quad (\text{since } S\in \mathcal{Z}_r^{(0,n-1)})$$

$$= b;$$

and so S belongs to $\mathcal{N}_n$. To prove the reverse containment, take any $S \in \mathcal{N}_n$. Then to show that S belongs to both $\mathcal{Z}_r^{(0,n-1)}$ and $\mathcal{Z}_1^{(n-1,0)}$, take any elements $a = a_1a_2 \ldots a_{n-1}$, x and y.

Now,

$$xa = x\, a_1a_2 \ldots a_{n-1} = (yx)a_1a_2 \ldots a_{n-1} = yxa$$

and

$$ax = a_1a_2 \dots a_{n-1}\, x = a_1a_2 \dots a_{n-1}(xy) = axy.$$

Hence, S belongs to $\mathcal{Z}_r^{(0,n-1)} \cap \mathcal{Z}_l^{(n-1,0)}$, proving that (c'') holds.

(d'') Finally, to prove this result, we will consider each one of the three cases independently, namely: $i+k = n$, $i+k > n$ and $i+k < n$.

Case i + k = n: We will show that as i and k run through the set $\{1,2, \dots, n\}$, and provided that $i+k = n$, the equality below holds:

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n-k,0)} \cap \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)} = \mathcal{Z}_l^{(i-1,k)}$$

Take any semigroup S in $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n-k,0)} \cap \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$, and any elements $a = a_1a_2 \dots a_{i-1}$, $b = b_1b_2 \dots b_k$, x and y.
Now, since $i+k = n$, we have $k = n-i$ and $i = n-k$, and so

$$axb = (ax)b = (ax)\,[b\,(axy)b] = [(ax)\,(b)\,(ax)]\,(yb) = (ax)\,(yb) = axyb.$$

We have the second equality since S belongs to $(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n-k,0)}$, and the fourth equality since S belongs to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$. Hence S belongs to $\mathcal{Z}_l^{(i-1,k)}$. To prove the reverse containment, take any semigroup S in $\mathcal{Z}_l^{(i-1,k)}$. Then to show that S belongs to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$, take any elements $a = a_1a_2 \dots a_i$, $b = b_1b_2 \dots b_i$ and $c = c_1c_2 \dots c_{n-i}$ of S.
Now, again since $k = n-i$ and $i = n-k$,

$$ac = (a_1a_2 \dots a_i)\,(c_1c_2 \dots c_{n-i}) = (a_1a_2 \dots a_{i-1})\,[(a_i)\,(ba)]\,(c_1c_2 \dots c_{n-i}) = abac;$$

and so S belongs to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$. To show that S belongs also to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n-k,0)}$, take any elements $s = s_1s_2 \dots s_{n-k}$, $t = t_1t_2 \dots t_k$, $u = u_1u_2 \dots u_k$. Then

$$ts = (t_1t_2 \dots t_{k-1})\,(t_k)\,(s_1 \dots s_{n-k}) = (t_1t_2 \dots t_{k-1})\,[(t_k)\,(ut)]\,(s_1 \dots s_{n-k}) = tuts.$$

We have thus shown that S belongs to $\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n-k,0)}$. Hence, the equality holds for the case $k+i = n$.

Case i + k < n: We will prove that as i and k run through the set $\{1,2, \dots, n\}$, and provided that $i + k < n$, the following equality holds:

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(n-k,0)} \cap \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^i\}^{(0,n-i)} = \mathcal{N}_{n-(i+k)}^{(i,k)}\ .$$

To prove this, take any semigroup $S \in \mathcal{N}_{n-(i+k)}{}^{(i,k)}$ and any elements $a = a_1a_2 \ldots a_{n-k}$, $b = b_1b_2 \ldots b_k$ and $c = c_1c_2 \ldots c_k$. We will show that S belongs to $\{(Z_l \vee Z_r)^k\}^{(n-k,0)}$.

Then

$$ab = (a_1a_2 \ldots a_{n-k})\,(b_1b_2 \ldots b_k)$$

$$= (a_1a_2 \ldots a_i)\,[(a_{i+1}a_{i+2} \ldots a_{n-k})]\,(b_1b_2 \ldots b_k) \qquad \text{(since } i < n\text{-}k)$$

$$= (a_1a_2 \ldots a_i)\,[(a_{i+1}a_{i+2} \ldots a_{n-k})\,(bc)\,]\,(b_1b_2 \ldots b_k)$$

$$= abcb.$$

We have the second last equality since S belongs in $\mathcal{N}_{n-(i+k)}{}^{(i,k)}$. We have thus shown that S also belongs to $\{(Z_l \vee Z_r)^k\}^{(n-k,0)}$. To show that S also belongs to $\{(Z_l \vee Z_r)^i\}^{(0,n-i)}$, take any elements $a = a_1a_2 \ldots a_i$, $b = b_1b_2 \ldots b_i$ and $c = c_1c_2 \ldots c_{n-i}$.

Now,

$$ac = (a_1a_2 \ldots a_i)\,(c_1c_2 \ldots c_{n-i})$$

$$= (a_1a_2 \ldots a_i)\,(c_1c_2 \ldots c_{n-i-k})\,(c_{n-i-k+1}c_{n-i-k+2} \ldots c_{n-i}) \qquad \text{(since } i+k < n)$$

$$= (a_1a_2 \ldots a_i)\,[(ba)\,(c_1c_2 \ldots c_{n-i-k})]\,(c_{n-i-k+1}c_{n-i-k+2} \ldots c_{n-i})$$

$$= abac;$$

and so S belongs also to $\{(Z_l \vee Z_r)^i\}^{(0,n-i)}$. These containments together imply that S belongs to $\{(Z_l \vee Z_r)^k\}^{(n-k,0)} \cap \{(Z_l \vee Z_r)^i\}^{(0,n-i)}$.

To prove the reverse containment, take any semigroup S in $\{(Z_l \vee Z_r)^k\}^{(n-k,0)} \cap \{(Z_l \vee Z_r)^i\}^{(0,n-i)}$ and any elements $a = a_1a_2 \ldots a_i$, $b = b_1b_2 \ldots b_k$, $c = c_1c_2 \ldots c_{n-(i+k)}$ and $d = d_1d_2 \ldots d_{n-(i+k)}$. We will show that S belongs to the variety $\mathcal{N}_{n-(i+k)}{}^{(i,k)}$.

Now, since $i+k < n$,

$$acb = (ac)\,b = (ac)\,[b\,(ad)^{k+i}\,b] = [a\,(cb\,(ad)^{k+i-1})\,a]\,(db) = adb.$$

We have the second equality since S belongs to $\{(Z_l \vee Z_r)^k\}^{(n-k,0)}$, and the third equality since S belongs to $\{(Z_l \vee Z_r)^i\}^{(0,n-i)}$; and that completes the proof for the case $i+k < n$.

Case $i + k > n$: We will show that as i and k run through the set $\{1,2, \ldots, n\}$ such that $i + k > n$, the following equality holds:

$$\{(Z_l \vee Z_r)^k\}^{(n-k,0)} \cap \{(Z_l \vee Z_r)^i\}^{(0,n-i)} = \{(Z_l \vee Z_r)^{i+k-n}\}^{(n-k,n-i)}.$$

Take any semigroup S in $\{(Z_l \vee Z_r)^k\}^{(n-k,0)} \cap \{(Z_l \vee Z_r)^i\}^{(0,n-i)}$, and any elements

$a = a_1a_2 \ldots a_{n-k}$, $b = b_1b_2 \ldots b_{n-i}$, $c = c_1c_2 \ldots c_{i+k-n}$ and $d = d_1d_2 \ldots d_{i+k-n}$.

We will show that S belongs to $\{(Z_l \vee Z_r)^{i+k-n}\}^{(n-k,n-i)}$. Since $i+k > n$, by assumption, we have that

$$\begin{aligned}
acb &= (a_1a_2 \ldots a_{n-k})\,(c_1c_2 \ldots c_{i+k-n})\,(b_1b_2 \ldots b_{n-i}) \\
&= a(cb) \\
&= a\,[(cb)\,(dacd)(cb)] \quad (\text{since } S \in \{(Z_l \vee Z_r)^k\}^{(n-k,0)}) \\
&= [(ac)\,(bd)\,(ac)]\,(dcd) \\
&= (ac)\,(dcb) \qquad\qquad (\text{since } S \in \{(Z_l \vee Z_r)^i\}^{(0,n-i)}) \\
&= acdcb;
\end{aligned}$$

and so S belongs to $\{(Z_l \vee Z_r)^{i+k-n}\}^{(n-k,n-i)}$. Now, to prove the reverse containment, take any semigroup S in $\{(Z_l \vee Z_r)^{i+k-n}\}^{(n-k,n-i)}$ and any elements $a = a_1a_2 \ldots a_{n-k}$, $b = b_1b_2 \ldots b_k$ and $c = c_1c_2 \ldots c_k$.

Now, since $i + k > n$, let $p = b_1b_2 \ldots b_{i+k-n}$, $q = b_{i+k-n+1}b_{i+k-n+2} \ldots b_k$ and $b = pq$. Then

$$\begin{aligned}
ab &= (a_1a_2 \ldots a_{n-k})\,(b_1b_2 \ldots b_k) = a\,pq \\
&= a\,[p\,(qc)\,p]\,q \qquad (\text{since } S \in \{(Z_l \vee Z_r)^{i+k-n}\}^{(n-k,n-i)}) \\
&= a\,(pq)\,c\,(pq) \\
&= abcb;
\end{aligned}$$

and so we have proved that $S \in \{(Z_l \vee Z_r)^k\}^{(n-k,0)}$. Next, we will show that S belongs also to $\{(Z_l \vee Z_r)^i\}^{(0,n-i)}$. Take any elements $s = s_1s_2 \ldots s_{n-i}$, $t = t_1t_2 \ldots t_i$ and $r = r_1r_2 \ldots r_i$.

As before, since $i+k > n$, let $w = (t_1t_2 \ldots t_{n-k})$, $v = (t_{n-k+1}t_{n-k+2} \ldots t_i)$ and $t = wv$. Then

$$\begin{aligned}
ts &= (t_1t_2 \ldots t_i)\,(s_1s_2 \ldots s_{n-i}) \\
&= (t_1t_2 \ldots t_{n-k})\,(t_{n-k+1}t_{n-k+2} \ldots t_i)\,(s_1s_2 \ldots s_{n-i}) = wvs \\
&= w\,[\,v\,(rw)\,v]\,s \qquad (\text{since } S \in \{(Z_l \vee Z_r)^{i+k-n}\}^{(n-k,n-i)}) \\
&= (wv)\,r\,(wv)\,s = t\,r\,t\,s;
\end{aligned}$$

and so S belongs to $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$. We have thus shown that S belongs to $\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^k\}^{(n-k,0)} \cap \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^i\}^{(0,n-i)}$, and that completes the proof. ❑

We have shown in this chapter that each P(i) forms a sublattice. Hence the skeleton $\mathcal{L}_S((\mathcal{Z}_1 \vee \mathcal{Z}_r)^n)$ is a pairwise disjoint union of such sublattices. The next result is an isomorphic construction of these planes, in order to understand better their structure.

Theorem 7:6.8 *For each positive integer* $n \geq 1$, *let* $X_n = \{0,1,2,3, \ldots n\}$ *and define a pair of binary operations* $\oplus$ *and* $\otimes$ *on the Cartesian product*

$$X_n \times X_n = \{(x,y) : x,y \in X_n\}$$

as follows:

$$(a,b) \oplus (e,f) = (\max\{a,e\}, \max\{b,f\})$$

and

$$(a,b) \otimes (e,f) = (\min\{a,e\}, \min\{b,f\}).$$

Then the lattice $(X_n \times X_n ; \oplus,\otimes)$ *is isomorphic to the plane* P(n).

Proof. It is easy to verify that $(X_n \times X_n; \oplus,\otimes)$ forms a lattice, and that P(n) form a lattice is a consequence of Theorems 7:6.5 and Theorem 7:6.6. The following elements of $X_n \times X_n$ form a chain analogous to a chain formed by the P(n)-join irreducible varieties on P(n), namely:

$$\{(i,0) : i = 0,1,2, \ldots, n-1\},$$

where

$$(0,0) \leq (1,0) \leq (2,0) \leq \ldots \leq (n-1,0) \leq (n,0)$$

The identification $(i,0) \mapsto \mathcal{N}_{n-i}{}^{(i,0)}$ for $i = 0,1,2 \ldots n-1$, and $(i,0) \mapsto \mathcal{Z}_1{}^{(n-1,0)}$ if $i = n$, allows us to see that the above chain and the following chain of P(n)-meet irreducible varieties on P(n) are isomorphic:

$$\mathcal{N}_n \subseteq \mathcal{N}_{n-1}{}^{(1,0)} \subseteq \mathcal{N}_{n-2}{}^{(2,0)} \subseteq \mathcal{N}_{n-3}{}^{(3,0)} \subseteq \ldots \subseteq \mathcal{N}_1{}^{(n-1,0)} \subseteq \mathcal{Z}_1{}^{(n-1,0)}.$$

Dually, the chain

$$(0,0) \leq (0,1) \leq (0,2) \leq \ldots \leq (0,n-1) \leq (0,n),$$

formed by members of the set $\{(0,j) : j = 0,1,2, \ldots, n-1\}$, and the following chain of P(n)-meet irreducible varieties on P(n):

$$\mathcal{N}_n \subseteq \mathcal{N}_{n-1}{}^{(0,1)} \subseteq \mathcal{N}_{n-2}{}^{(0,2)} \subseteq \mathcal{N}_{n-3}{}^{(0,3)} \subseteq \ldots \subseteq \mathcal{N}_1{}^{(0,n-1)} \subseteq \mathcal{Z}_r{}^{(0,n-1)},$$

are isomorphic under the identification: $(0,j) \mapsto \mathcal{N}_{n-j}^{(0,j)}$ for $j = 0,1,2...n-1$, and $(0,j) \mapsto \mathcal{Z}_1^{(0,n-1)}$ if $j = n$.

We note also that every member of $(X_n \times X_n; \oplus,\otimes)$ can be expressed uniquely as a $\oplus$-product of some members of these chains. In fact, for any take any (x,y) on $X_n \times X_n$, we see that

$$(x,y) = (x,0) \oplus (0,y).$$

The chains described above generate the entire lattice $(X_n \times X_n; \oplus,\otimes)$, in the sense that every member of $(X_n \times X_n; \oplus,\otimes)$ can be expressed as a $\oplus$-product of these chains. Similarly, as we saw in Corollary 7:6.4, every variety on P(n) can be expressed uniquely as a join of some P(n)-join irreducible varieties. Hence, the above identification (isomorphism of chains) can be extended to an isomorphism between $(X_n \times X_n; \oplus,\otimes)$ and P(n).

More formally, we define the isomorphism:

$$\Phi_n : (X_n \times X_n; \oplus,\otimes) \longrightarrow P(n),$$

by the rule that for each (x,y,n) in $(X_n \times X_n; \oplus,\otimes)$

$$(x,y)\Phi_n = (x,0)\Phi_n \vee (0,y)\Phi_n,$$

where $\vee$ denotes the varietal join on P(n),

$$(i,0)\Phi_n = \begin{cases} \mathcal{Z}_r^{(n-1,0)} & , \text{ if } i = n \\ \mathcal{N}_{n-i}^{(i,0)} & , \text{ if } i = 1,2,\ldots,n-1 \end{cases}$$

and

$$(0,i)\Phi_n = \begin{cases} \mathcal{Z}_r^{(0,n-1)} & , \text{ if } i = n \\ \mathcal{N}_{n-i}^{(0,i)} & , \text{ if } i = 1,2,\ldots,n-1 \end{cases}$$

❑

Theorem 7:6.9 *For every positive integer n, and any ordered pair (i,j) of non negative integers, the map* $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$ *preserves varietal joins and meets on P(n).*

Proof. We know from Theorem 2:2.6 that the map $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$ preserves all varietal meets, and that it is one-to-one. Although we do not know whether or not it preserves varietal joins in general, we are able to say here that it does on P(n).

Recall that $P(n) = \{ \mathcal{V} \in \wp : \mathcal{N}_n \subseteq \mathcal{V} \subseteq (\mathcal{Z}_1 \vee \mathcal{Z}_r)^n \}$. The image of this interval under the map $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$ is

$$P(n)^{(i,j)} = \{ \mathcal{V} \in \wp : \mathcal{N}_n^{(i,j)} \subseteq \mathcal{V} \subseteq \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^n\}^{(i,j)} \},$$

which lies on the plane P(n+i+j). We know that the map is one to one, that it maps P(n) onto $P(n)^{(i,j)}$, and that it preserves that lattice structure of P(n) (since varietal meets are preserved). Since $P(n)^{(i,j)}$ forms a sublattice of the plane P(n+1+i+j) (from Figure 7:6.1), it forces the map to become a lattice isomorphism in view of Lemma 0:4.20. Hence, it preserves varietal joins and that completes the proof. ❑

An alternative proof would be to tediously check that for any two varieties A and B on P(n), the join of their images is equal to the image of their join. But that approach, though it works, would take a tremendous amount of time and space. Moreover, it would be useful to find an alternative proof for Theorem 7:6.9 without making use of the recursive relations proved in this thesis, as it would mean that we could then use Theorem 7:6.9 to reduce some of the long proofs that were required for proving the recursive relations in Sections 7:2, 7:3 and 7:4.

Problem 7:6.10 *Is it also true that the map $\mathcal{V} \mapsto \mathcal{V}^{(i,j)}$ preserves varietal joins and meets of structurally trivial varieties in general?* ❑

7:7 SOME EXAMPLES OF PLANES AND THEIR USES

In this section we will put together the information we have gathered thus far about the planes, and learn what we can from the nice lattice structures they form. We saw in Section 7:6 that the skeleton $\mathcal{L}_S((\mathcal{Z}_l \vee \mathcal{Z}_r)^n)$ is a pairwise disjoint union of the planes P(1), P(2) . . . . P(n), and each of these planes is closed under taking varietal joins and meets. Therefore, the skeleton $\mathcal{L}_S((\mathcal{Z}_l \vee \mathcal{Z}_r)^n)$ is a disjoint union of a family of sublattices of the lattice of all semigroup varieties.

Theorem 7:7.1 *The skeleton of the lattice $\mathcal{L}((\mathcal{Z}_l \vee \mathcal{Z}_r)^n)$ is a pairwise disjoint unions of n planes, each of which forms a sublattice, and their union forms an inverted pyramid shown on Figure 7:7.3.* ❑

It is not clear at this stage whether or not the entire collection $\mathcal{L}_S((\mathcal{Z}_l \vee \mathcal{Z}_r)^n)$ of varieties forms a sublattice, that is, we do not know whether or not that $\mathcal{L}_S((\mathcal{Z}_l \vee \mathcal{Z}_r)^n)$ is closed under taking varietal joins and varietal meets. The solutions to the following problem would help to answer this question.

Problem 7:7.2 *Let $i \le n$ and $j \le n$. Is the partially ordered set $\mathcal{L}_S((\mathcal{Z}_l \vee \mathcal{Z}_r)^n)$ closed under taking:*

(a) joins of P(i)-join irreducible and P(j)-join irreducible varieties?

(b) meets of P(i)-meet irreducible and P(j)-meet irreducible varieties?

❑

Problem 7:7.3 *Is the plane P(n) convex, in the sense that it consists of all semigroup varieties in the interval $\mathcal{N}_n \subseteq \mathcal{V} \subseteq (\mathcal{Z}_l \vee \mathcal{Z}_r)^n$? Recall, that by definition P(n) consists only of the members of $\wp$ which are in the above interval.* ❑

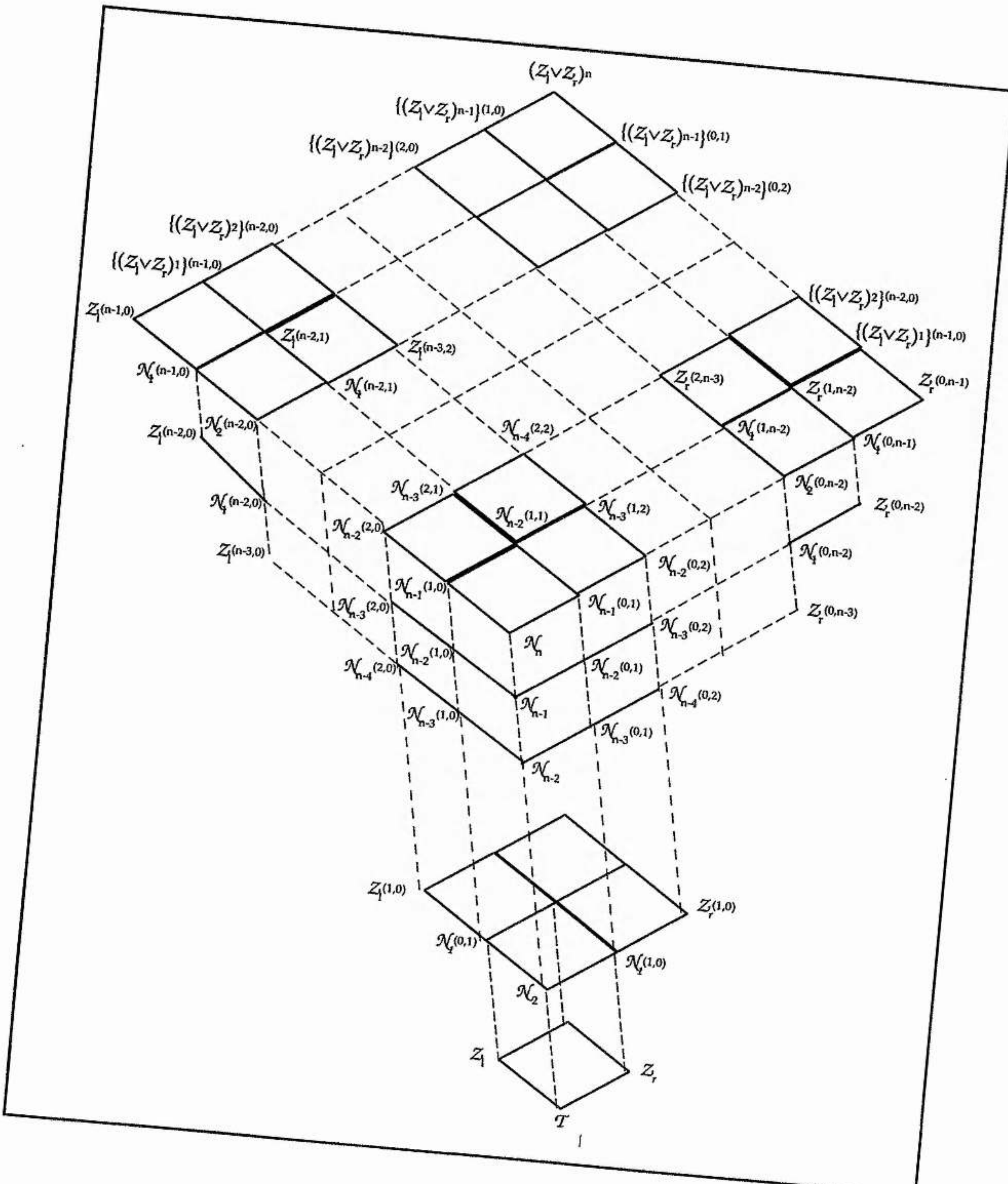


Figure 7:7.4 The skeleton of $\mathcal{L}((Z_l \vee Z_r)_n)$

Theorem 7:7.5 *Let $n \geq 1$. The variety of all n-inflations of rectangular bands is as follows:*

$$(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(n-1,0)} \cap (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,n-1)} = \mathcal{N}_n \vee (\mathcal{Z}_l \vee \mathcal{Z}_r) = \mathcal{N}_{n-2}^{(1,1)}$$

and the lattice intervals $[\mathcal{T}, (\mathcal{Z}_l \vee \mathcal{Z}_r)]$ *and* $[\mathcal{N}_n, \mathcal{N}_{n-2}^{(1,1)}]$ *are isomorphic. Moreover, the variety of all n-inflations of rectangular bands can be represented as an equational class in the following two different ways:*

$$[(x_1x_2\ldots x_{n-1})aba = (x_1x_2\ldots x_{n-1})a,\ a(x_1x_2\ldots x_{n-1}) = aba\,(x_1x_2\ldots x_{n-1})]$$

and

$$[a(x_1x_2\ldots x_{n-2})b = a(x_1x_2\ldots x_{n-2})b]$$

Proof. Since rectangular bands are regular, we have by Theorem 6:2.6 that the variety $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(n-1,0)} \cap (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,n-1)}$ consists of all n-inflations of semigroups in $(\mathcal{Z}_l \vee \mathcal{Z}_r)$. In fact, by that same result,

$$(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(n-1,0)} \cap (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,n-1)} = \mathcal{N}_n \vee (\mathcal{Z}_l \vee \mathcal{Z}_r).$$

The meet $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(n-1,0)} \cap (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,n-1)}$ sits on P(n) since it (the plane P(n)) is closed under varietal meets (from Theorem 7:6.6). In fact, from Figure 7:6.1 we have

$$(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(n-1,0)} \cap (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,n-1)} = \mathcal{N}_{n-2}^{(1,1)}.$$

It follows from these equalities that

$$\mathcal{N}_{n-2}^{(1,1)} = \mathcal{N}_n \vee (\mathcal{Z}_l \vee \mathcal{Z}_r).$$

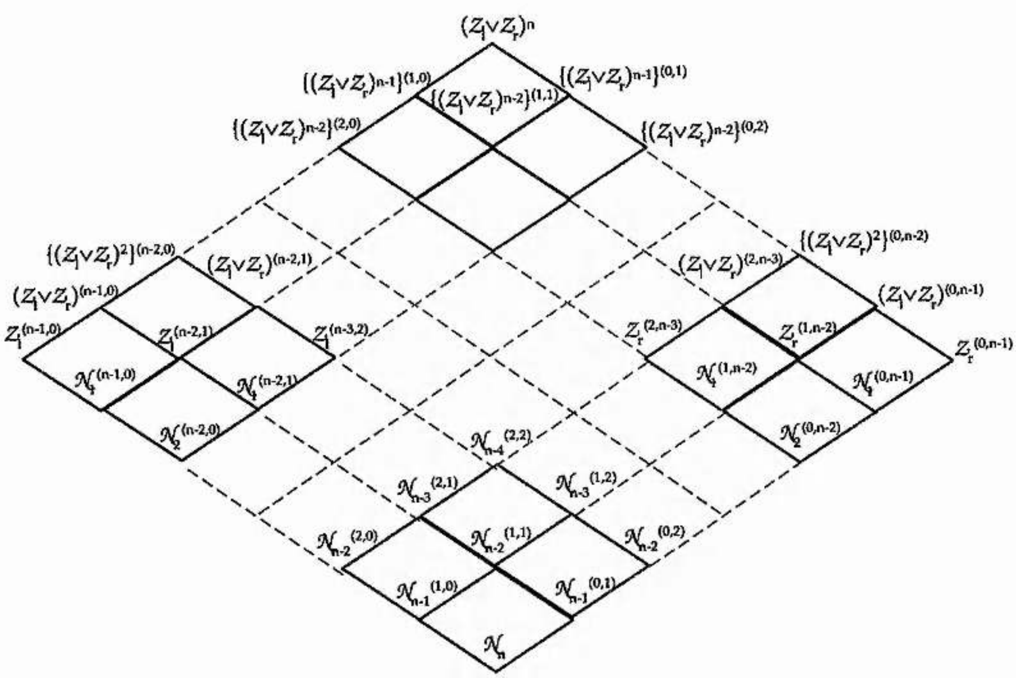
The isomorphism of the lattice intervals $[\mathcal{T}, (\mathcal{Z}_l \vee \mathcal{Z}_r)]$ and $[\mathcal{N}_n, \mathcal{N}_{n-2}^{(1,1)}]$ is a consequence of Theorem 6:3.2; the lattice interval $[\mathcal{T}, (\mathcal{Z}_l \vee \mathcal{Z}_r)]$ is precisely the four element closed diamond P(1) (see Figure 7:5.9), and so $[\mathcal{N}_n, \mathcal{N}_{n-2}^{(1,1)}]$ is also a closed diamond. The equality of the given equational classes follow from (7:1.10), (7:1.4) and Theorem 6:2.1, that $\mathcal{N}_{n-2}^{(1,1)}$ is determined by the identity

$$a(x_1x_2\ldots x_{n-2})b = a(x_1x_2\ldots x_{n-2})b$$

while the variety $(\mathcal{Z}_l \vee \mathcal{Z}_r)^{(n-1,0)} \cap (\mathcal{Z}_l \vee \mathcal{Z}_r)^{(0,n-1)}$ is determined by the pair

$$(x_1x_2\ldots x_{n-1})aba = (x_1x_2\ldots x_{n-1})a,\ a(x_1x_2\ldots x_{n-1}) = aba\,(x_1x_2\ldots x_{n-1}) \qquad \square$$

Example 7:7.6 We demonstrate how nicely and compactly, the lattice diagram of P(n) below contains information about certain varietal relations on P(n), which otherwise, would be difficult to observe or prove.



The following information can be deduced easily from this lattice:

(A) Each variety on P(n) is determined by a single identity, but for those varieties which are not P(n)-meet irreducible, they can also be determined by at least one pair of identities. In fact, if a variety $\mathcal{V}$ on P(n) can be expressed as a meet of two other varieties, say $\mathcal{U}$ and $\mathcal{W}$, then $\mathcal{V}$ is determined by the set of identities consisting of the one which determine $\mathcal{U}$ and the one which determines $\mathcal{W}$. It is an exercise to determine all possible distinct pairs of identities which determines $\mathcal{V}$, by determining all possible ways of expressing $\mathcal{V}$ as the meet of two others.

(B) The lattice diagram of P(n) generalises all the recursive relations proved in some earlier sections of this chapter, involving meets and joins. The above diagrams beautifully describe all possible closed diamonds which can be formed on P(n). It is tedious, and not worth the effort and the space, to write all possible recursive relations, but there are some of which that can be compactly and nicely expressed:

(B1) If (i,j) and (r,s) are ordered pairs of non-negative integers, with k>0 and m>0 such that i+j+k = n = r+s+m, then

$$\mathcal{N}_k^{(i,j)} \bigcap \mathcal{N}_m^{(r,s)} = \mathcal{N}_w^{(p,q)},$$

where p = min{i,r}, q = min {j,s} and w = n - (p+q).

We see that (B1) is a generalisation of Theorem 7:3.4, while the following is a generalisation of Theorem 7:2.6.

(B2) If (i,j) and (r,s) are ordered pairs of non-negative integers, with k>0 and w>0, such that i+j+k = n = r+s+m, then

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(i,j)} \vee \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^m\}^{(r,s)} = \{(\mathcal{Z}_l \vee \mathcal{Z}_r)^w\}^{(p,q)},$$

where p = min{i,r}, q = min {j,s} and w = n - (p+q).

Therefore, the set of varieties:

$$\{(\mathcal{Z}_l \vee \mathcal{Z}_r)^k\}^{(i,j)} : k \geq 1, i \geq 0, j \geq 0, i+j+k = n\}$$

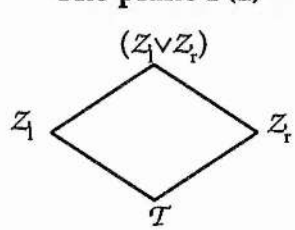
is closed under taking varietal joins, while the set

$$\{\mathcal{N}_k^{(i,j)} : k \geq 1, i \geq 0, j \geq 0, i+j+k = n\}$$

is closed under taking varietal meets. There are, of course many other relations on P(n), which have complicated description and are therefore left out, but two of which are expressed as Theorems 7:7.8 and 7:7.9 below. ❑

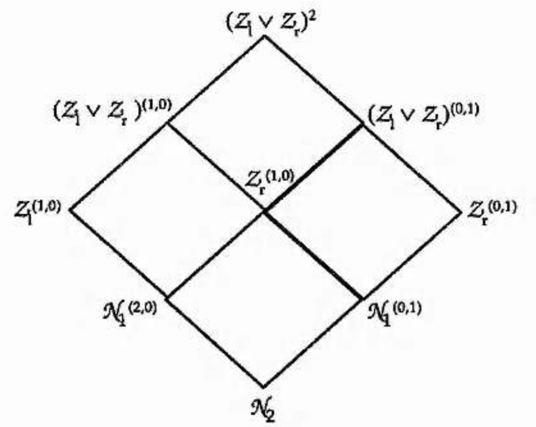
Example 7:7.7 For the author's own future benefit, and for some readers who might wish to see particular examples of these planes, we have written out completely, the lattice structure of the planes P(1), P(2), P(3), P(4) and P(5):

The plane P(1)



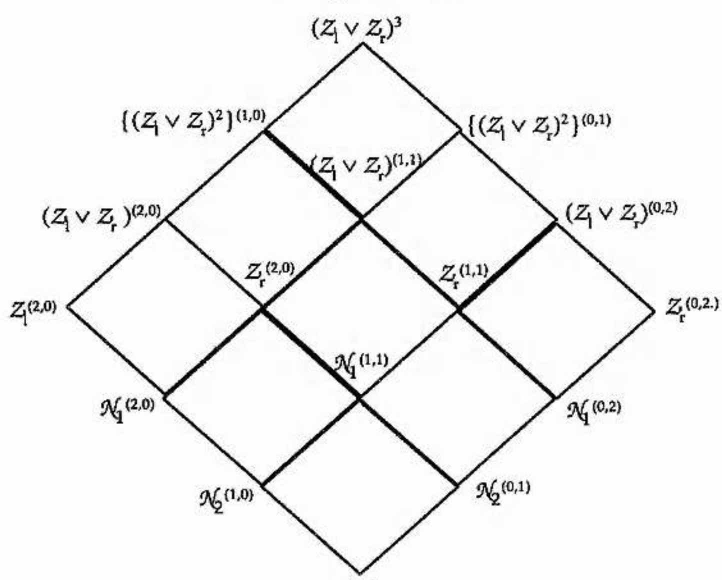
P(1) consists of all varieties of rectangular bands and is well known.

The plane P(2)



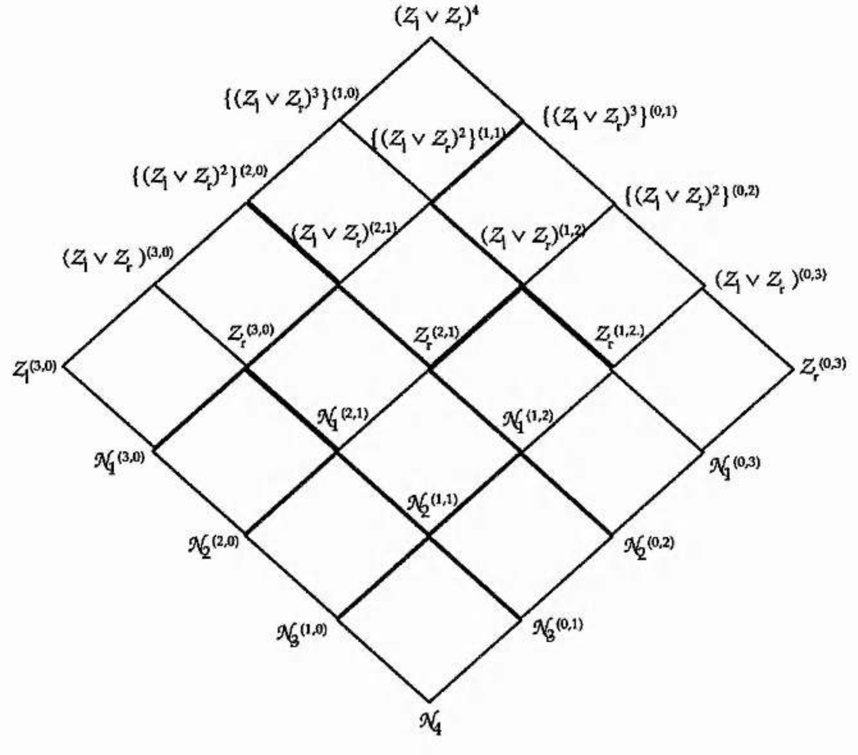
This lattice interval was first determined by Melnik (1970) (Figures 0:4.15 and 7:5.9) and later by Petrich (1974). Of course, in our lattice diagram of P(2) above, we have labelled the varieties differently, and have offered an alternative (singleton) set of identities for each variety on P(2).

The plane P(3)



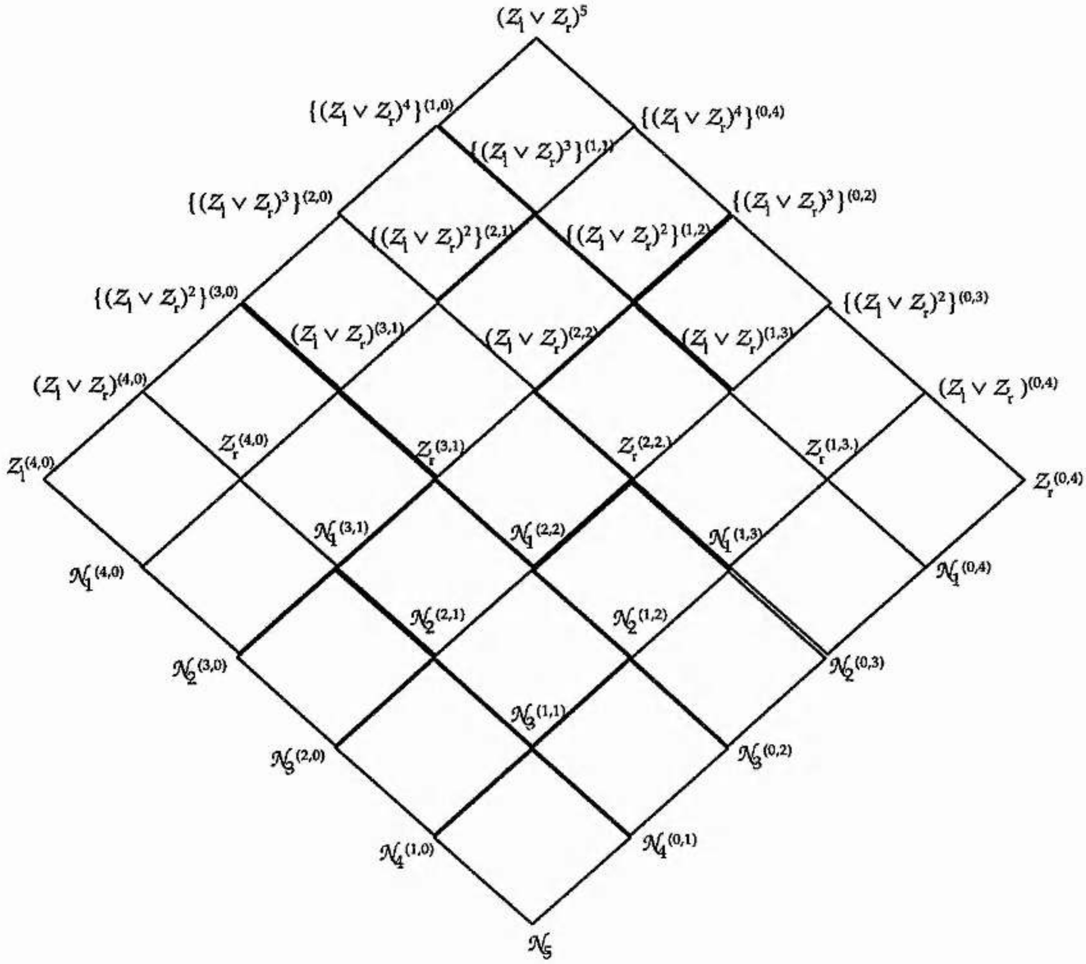
For all P(k), for k≥3, the lattice structures of P(k) have not appeared in the literature, at least from what the author has had access to.

The plane P(4)



At this stage, the only known convex diamond on P(n) is the one formed by the varieties $\{\mathcal{N}_n, \mathcal{N}_{n-2}^{(1,1)}, \mathcal{N}_{n-1}^{(1,0)}, \mathcal{N}_{n-1}^{(0,1)}\}$, in view of Theorem 7:7.5. By this we mean that we do not know, if as pointed out in Problem 7:7.4, if there are other varieties of semigroups which occur between two members of P(n) but which do not sit on P(n). In the case of P(4), the diamond formed by $\{\mathcal{N}_4, \mathcal{N}_2^{(1,1)}, \mathcal{N}_3^{(1,0)}, \mathcal{N}_3^{(0,1)}\}$ is convex.

The plane P(5)



Each one of these plane is closed under varietal joins and varietal meets, thus forming a sublattice of the lattice of all semigroup varieties. The identities determining each one of these varieties can be determined using the rules described in Section 7:1, namely from the list (7:1.2) - (7:1.14). ❑

The following result generalises Theorem 7:3.3, making use of the fact (proved in Section 7:6) that P(n) is closed under varietal joins, whose lattice structure is as shown on Figure 7:6.1. Proving these results otherwise, would involve a tremendous amount of time, effort and space.

Theorem 7:7.8 *Let (i,j) and (r,s) be ordered pairs of non-negative integers, with k>0 and t>0 such that i+j+k = n = r+s+t.*
Then

$$\mathcal{N}_k^{(i,j)} \vee \mathcal{N}_t^{(r,s)} \quad = \quad \begin{cases} \mathcal{Z}_1^{(u-1,v)} \quad , & if \ \ u+v = n \\ \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{u+v-n}\}^{(n-u,n-v)} \quad , & if \ \ u+v > n \\ \mathcal{N}_{n-(u+v)}^{(u,v)} \quad , & if \ \ u+v < n \end{cases}$$

where $u = \max\{i,r\}$ *and* $v = \max\{j,s\}$.

Proof. We first observe from Figure 7:6.1 that the varieties $\mathcal{N}_k^{(i,j)}$ and $\mathcal{N}_t^{(r,s)}$ can be expressed as joins as follows:

$$\mathcal{N}_k^{(i,j)} = \mathcal{N}_{k+j}^{(i,0)} \vee \mathcal{N}_{k+i}^{(0,j)} \quad \text{and} \quad \mathcal{N}_t^{(r,s)} = \mathcal{N}_{t+s}^{(r,0)} \vee \mathcal{N}_{t+r}^{(0,s)} .$$

Now,

$$\mathcal{N}_k^{(i,j)} \vee \mathcal{N}_t^{(r,s)} = (\mathcal{N}_{k+j}^{(i,0)} \vee \mathcal{N}_{k+i}^{(0,j)}) \vee (\mathcal{N}_{t+s}^{(r,0)} \vee \mathcal{N}_{t+r}^{(0,s)})$$

(by incorporating the above two equalities)

$$= (\mathcal{N}_{k+j}^{(i,0)} \vee \mathcal{N}_{t+s}^{(r,0)}) \vee (\mathcal{N}_{k+i}^{(0,j)} \vee \mathcal{N}_{t+r}^{(0,s)})$$

(since varietal join is commutative and associative)

$$= \mathcal{N}_{n-u}^{(u,0)} \vee \mathcal{N}_{n-v}^{(0,v)}, \text{ where } u = \max\{i,r\} \text{ and } v = \max\{j,s\}$$

(by the chain formation of P(n)-join irreducible varieties)

The conclusion then follows easily from Equality (a') in the proof of Theorem 7:6.5. ❑

Theorem 7:7.9 *Let (i,j) and (r,s) be ordered pairs of non-negative integers, with k>0 and t>0 such that i+j+k = n = r+s+t.*
Then

$$\{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^k\}^{(i,j)} \ \cap \ \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^t\}^{(r,s)} \quad = \quad \begin{cases} \mathcal{Z}_1^{(n-v-1,n-u)} \quad , & \text{if } \ u+v = n \\ \{(\mathcal{Z}_1 \vee \mathcal{Z}_r)^{n-(u+v)}\}^{(u,v)} \ , & \text{if } \ u+v < n \\ \mathcal{N}_{(u+v)-n}^{(n-v,n-u)} \ , & \text{if } \ u+v > n \end{cases}$$

where $u = \max\{i,r\}$ *and* $v = \max\{j,s\}$.

Proof. First we observe from Figure 7:6.1 that the varieties $\{(Z_1 \vee Z_r)^k\}^{(i,j)}$ and $\{(Z_1 \vee Z_r)^t\}^{(r,s)}$ can be expressed as meets as follows:

$$\{(Z_1 \vee Z_r)^k\}^{(i,j)} = \{(Z_1 \vee Z_r)^{k+j}\}^{(i,0)} \bigcap \{(Z_1 \vee Z_r)^{k+i}\}^{(0,j)}$$

and

$$\{(Z_1 \vee Z_r)^t\}^{(r,s)} = \{(Z_1 \vee Z_r)^{t+s}\}^{(r,0)} \bigcap \{(Z_1 \vee Z_r)^{t+r}\}^{(0,s)} .$$

Now,

$\{(Z_1 \vee Z_r)^k\}^{(i,j)} \bigcap \{(Z_1 \vee Z_r)^t\}^{(r,s)}$

$= [\{(Z_1 \vee Z_r)^{k+j}\}^{(i,0)} \bigcap \{(Z_1 \vee Z_r)^{k+i}\}^{(0,j)}] \bigcap [\{(Z_1 \vee Z_r)^{t+s}\}^{(r,0)} \bigcap \{(Z_1 \vee Z_r)^{t+r}\}^{(0,s)}]$

(by incorporating the above two equalities)

$= [\{(Z_1 \vee Z_r)^{k+j}\}^{(i,0)} \bigcap \{(Z_1 \vee Z_r)^{t+s}\}^{(r,0)}] \bigcap [\{(Z_1 \vee Z_r)^{k+i}\}^{(0,j)} \bigcap \{(Z_1 \vee Z_r)^{t+r}\}^{(0,s)}]$

(since varietal meet is commutative and associative)

$= \{(Z_1 \vee Z_r)^{n-u}\}^{(n,0)} \bigcap \{(Z_1 \vee Z_r)^{n-v}\}^{(0,v)}]$, where $u = \max\{i,r\}$ and $v = \max\{j,s\}$.

(by the chain formation of P(n)-meet irreducible varieties)

The conclusions then follows from Lemma 7:6.7 (d"). ❑

The above result generalises Theorem 7:2.6.

Remark 7:7.10 The results proved so far in this thesis can be combined together to describe completely certain large sections (around the lower part) of the lattice of all semigroup varieties. But due to space and time limitations, we have had to stop here. The author feels strongly that the methods and results proved here could also be adapted easily to prove analogous results involving pseudovarieties and generalised varieties. Moreover, we also feel that these methods can be adapted to enrich the algebraic theories of: rings, near rings and semirings.

REFERENCES

ALMEIDA, J.,

1988a Power pseudovarieties of semigroups I, *Semigroup Forum* **33**, 357-373.

1986b Power pseudovarieties of semigroups II, *Semigroup Forum* **33**, 375-390.

ALMEIDA, J. and REILLY, N.R.,

1984 Generalised varieties of commutative and nilpotent semigroups, *Semigroup Forum* **26**, 77-98.

ASH, C.J.

1985 Pseudovarieties, generalised varieties and similarly defined classes", *J. Algebra* **92**, 104-115.

BIRJUKOV,A.P.,

1970 Varieties of idempotent semigroups, *Algebra i Logika* , 255-273.

BOGDANOVIC, S.,

1988 Inflation of union of groups, *Math. BECHK* **37**, 351 - 355.

BOGDANOVIC, S. and STAMENKOVIC, B.,

1988 Semigroups in which S^{n+1} is a semilattice of right groups (inflations of a semilattice of right groups), *Note Mat.* Vol.III-n.1, 155-172.

BOGDANOVIC, S. and SVETOZAR, M.,

1987 Inflation of semigroups, *Publ. Inst. Math.* Nouvelle serie, **41** (55), 63 - 73.

BURRIS, S. and NELSON, E.,

1971 Embedding the dual of Π_m in the lattice of all equational classes of commutative semigroups, *Proc. Amer. Math. Soc.* **30**, 37-39.

CLARKE, G.T.,

1981 Semigroup varieties of inflations of unions of groups, *Semigroup Forum* **23**, 311-319.

CLIFFORD, A.H. and PRESTON, G.B.,

1961 *The algebraic theory of semigroups*, Vol.1. American Mathematical Society, Providence, R.I..

1967 *The algebraic theory of semigroups Volume 2*, American Mathematical Society, Providence, R.I., .

COHN, P.M,
1965 *Universal Algebra*, Harper & Row.

DAVENPORT, P.M.,
1992 On power commutative semigroups, *Semigroup Forum* **44**, 9-19.

EDWARDS, C.C.,
1979 The structures of L-uniportant semigroups, *Semigroup Forum* **18**, 189-199.

EDWARDS, P.M,
1983 Eventually regular semigroups, *Bull. Austral. Math. Soc.* **23**, 23-38.

EILENBERG, S. and SCHÜTZENBERGER, M.P.,
1976 On pseudovarieties, *Adv. Math.* **19**, 413-418.

EVANS, T.,
1971 The lattice of semigroup varieties, *Semigroup Forum* **2**, 1-43.

FENNEMORE, C.,
1970 All varieties of bands, *Semigroup Forum*, **1**, 172-179.

FRALEIGH, J.B.,
1983 *A first course in abstract algebra (third edition)*, Addison-Wesley Publishing Company (1982)

GERHARD, J.A.,
1970 The lattice of Equational Classes of Idempotent Semigroups, *J. Algebra* **15** (No.2), 195-224.

1977(a) Semigroups with an idempotent power I: word problems, *Semigroup Forum* **14**, 137-141.

1977(b) Semigroups with an idempotent power II: lattice of equational classes of $(xy)^2=xy$", *Semigroup Forum* **14**, 375-388.

GERHARD, J.A. and PETRICH, M.,
1988 Certain characterisations of varieties of bands, *Proc. Edinburgh Math. Soc.* **31**, 301-319.

GOMES, G.M.S.,
1986 Some results on congruences of R-unipotent semigroups, *Semigroup Forum* **33**, 175-185.

GRÄTZER, G.,
1968 *Universal Algebra*, Princeton, N.J.

GREEN, J.A.,
1951 On the structure of semigroups, *Ann, Math.* **54**, 164-174.

HALL, T.E.,
1969 On regular semigroups whose idempotents form a subsemigroup, *Bull. Austral. Math. Soc.* **1**, 195-208.

1973 On regular semigroups, *J. Algebra* **24**, 1-24.

1989 Identities for existence varieties of regular semigroups", *Bull. Austral. Math. Soc.* **40**, 59-77.

1991 Regular semigroups: amalgamation and lattices of existence varieties, *Algebra Universalis* **28**, 79-102.

HIGGINS, P.M.,
1984 Saturated and Epimorphically closed varieties of semigroups, *Austral. Math. Soc.* Ser. A, **36** , 153-176.

1991 On eventually regular semigroups, *Proceedings of the conference on semigroups with applications, Oberwolfach* (Editors: J.M. Howie, W.D. Munn and H.J. Weinert), 170-189.

1992 *Techniques of semigroup theory,* Oxford University Press.

HOWIE, J.M.,
1964 The maximum idempotent separating congruen, *Proc. Edinburgh Math. Soc.* **14**, 71-79.

1976 *An introduction to semigroup theory*, Academic Press.

JIANG, Z.
1994 On (m,n)-commutative local (permutable) semigroup, *Semigroup Forum* **44**,

JONES, P.R.

1993 Monoids defined by $x^{n+1} = x$ are local, *Semigroup Forum* **47**, 318-326.

1989 Varieties generated by free completely regular semigroups, *Semigroup Forum* **38**, 269-282.

JONES, P.R. and TROTTER, P.G.,
1991(a) Joins of inverse semigroup varieties, *Internat. J. Algebra Comput.* **1**, 371-385.

1991(b) Joins of Inverse semigroup varieties and Band varieties, Technical Report No. 339 - January 1991.

KADOUREK, J. and SZENDREI, M.B.,
1990 A new approach in the theory of orthodox semigroups, *Semigroup Forum* **40**, 257-296.

KISIELEWICZ, A.,
All pseudovarieties of commutative semigroups, *Proceedings of the conference on semigroups with applications, Oberwolfach* (Editors: J.M. Howie, W.D. Munn and H.J. Weinert), 78-89.

KOPAMU, S.J.L.,
1991 *On engamorphic products and M(x)-varieties of Semigroups*, M.Sc. thesis, Monash University.

1994 Orthodox right quasi normal bands of groups, *Southeast Asian Bull. Math.*, **18** , 105-116.

1995(a) On semigroup species, *Comm. Algebra* (to appear).

1995(b) Varieties of structurally inverse semigroups, *Semigroup Forum* (submitted).

1995(c) Varieties formed by nilpotent extensions of rectangular groups, *Semigroup Forum* (submitted).

1995(d) The concept of structural regularity, *Portigal. Math.* (to appear)

1995(e) Varieties of structurally trivial semigroups I, *Semigroup Forum* (submitted).

KORJAKOV, I.,
1982 A sketch of the lattice of commutative nilpotent semigroup varieties, *Semigroup Forum* **24**, 285-317.

LALLEMENT, G.,
1967 Demi-groupes reguliers, *Ann. Mat. Pura Appl.* **77**, 47-130

LAWSON, M.V.
1989 An order theoretic characterisation of locally orthodox semigroups, *Semigroup Forum* **39**, 113-116.

LENDER, V.B.
1993 Retract and monomorphism monoids in semigroup varieties, *Semigroup Forum* **47**, 373-380.

MAL'TSEV,
1973 *Algebraic systems,* Springer-Verlag .

MEAKIN, J.,
1972(a) Congruences on orthodox semigroups II, *J. Austral. Math. Soc. Ser. A,* **11**, 259-266.

1972(b) The maximum idempotent separating congruence on regular semigroups, *Proceedings of Edinburgh Maths. Soc.* **17-18**, 159-163.

MELNIK,I.I.,
1970 On varieties and lattices of varieties of semigroups, *Investigations on Algebra, Saratov* **2**, 45-57, (Russian).

1971 On a family of semigroup varieties, *Izv. VUZ Matematika* **2**, 103-108, (Russian).

MUNN, W.D.,
1961 Pseudoinverses in semigroups, *Proc. Cambridge Philos. Soc.* **57**, 247-250.

1964 Brandt congruences on inverse semigroups, *Proc. London Maths. Soc.* **14**, 154-164.

1981 A class of irreducble matrix representation of an arbitrary inverse semigroup, *Proc. Glasgow Math. Soc.* **5**, 41-48.

NAMBOORIPAD, K.S.S.,
1980 The natural partial order order on a regular semigroup, *Proc. Edinburgh Math. Soc.* **23**, 249-260.

PASTIJN, F.,
1990 The lattice of completely regular semigroup varieties, *J. Austral. Math. Soc.* Ser. A, **323**, 104-152.

1991 Commuting fully invariant congruences on free completely regular semigroups, *Trans. Amer. Math. Soc.* **323**, 79-92.

PASTIJN, F. and YAN, X.,
1994(a) Varieties of semigroups and varieties of completely regular semigroups closed for certain extensions, *J. Algebra*, **163**, 777-794.

1994(b) Semigroup varieties closed for the Bruck extension, *Glasgow Math. J.* **36**, 371-380.

PETRICH, M. and REILLY, N.R.,
1984 The join of varieties of strict inverse semigroups and rectangular bands, *Glasgow Math. J.* **25** (1984), 59-74.

PETRICH, M.,
1973 *Introduction to semigroups*, Merill Columbus.

1974 All subvarieties of certain semigroup varieties, *Semigroup Forum* **7**, 104-152.

1975 Varieties of orthodox bands of groups, *Pacific J. Math.* **58**, 209-217.

1984 *Inverse Semigroups*, John Wiley and Sons.

PIN, J.-E. and D. Thérien,
1993 The bideterministic concatenation product, *Internat. J. Algebra Comput.* **3**, 535-555.

PIN, J.-E., STRAUBING, H. and THERIEN, D.,
Small varieties of groups, *J. Austral. Math. Soc.* **37**, 269-281.

POLLÀK, G.,
1985 On varieties of completely regular semigroups. I, *Semigroup Forum* **32** , 97-123.

1987 On varieties of completely regular semigroups. II, *Semigroup Forum* **36** , 253-284.

1989 A new example of limit variety, *Semigroup Forum* **38**, 283-303.

PUTCHA, M.S. and YAQUB, A.,
1971 Semigroups with permuting identities, *Semigroup Forum*, **1**, 68-73.

REILLY, N.R. and SCHEIBLICH, H.E.,
1967 Congruences on regular semigroups, *Pacific J. Math.* **23**, 349-360.

SAPIR, M. and VOLKOV, M.V.,
1994 On the joins of semigroup varieties with the variety of commutative semigroups, *Proc. Amer. Math. Soc.* **120**, 345-348.

SCHWABAUER, R.,
1969(a) A note on commutative semigroups, *Proc. Amer. Math. Soc.* **20**, 503-504.

1969(b) Commutative semigroup laws, *Proc. Amer. Math. Soc.* **20**, 591-595.

SHEVRIN, L.N., and VOLKOV,M.V.,
1985 Identities of semigroups, *Izvestiya VUZ. Matematika* **29**, 3-47.

VOLKOV, M.V.,
1982 An example of a limit variety, *Semigroup Forum* **24**, 319-324.

1994 Young Diagrams and the structure of the lattice of overcommutative varieties, *Transformation semigroups* , Essex University, (edited by P.M. Higgins)

VOLKOV, M.V. and ERSHOVA, T.A.,
1990 The lattice of all varieties of semigroups with completely regular square, *Semigroup Conference in honor of G.B. Preston*, Monash University, 308-309

YAMADA, M.,
1967 Regular semigroups whose idempotents satisfy permutative identities, *Pacific J. Math.* **21**, 371-392.

ZEITOUN, M.,
1995 The join of pseudovarieties of idempotent semigroups and locally trivial semigroups, *Semigroup Forum* **50**, 367-381.

ZHANG, S.,
1994 Completely regular semigroup varieties generated by Mal'cev products with groups, *Semigroup Forum* **48**, 180-192.

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